ECE276B: Planning & Learning in Robotics
Lecture 12: Model-free Control

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Model-free Generalized Policy Iteration

- **Model-based case**: our main tool for solving a stochastic infinite-horizon problem was Generalized Policy Iteration (GPI):
  - **Policy Evaluation**: Given π, compute V^π:
$$V^\pi(\mathbf{x}) = \ell(\mathbf{x}, \pi(\mathbf{x})) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\pi(\mathbf{x}))}\left[V^\pi(\mathbf{x}')\right], \quad \forall \mathbf{x} \in \mathcal{X}$$
  - **Policy Improvement**: Given V^π obtain a new policy π':
$$\pi'(\mathbf{x}) = \arg\min_{\mathbf{u} \in \mathcal{U}(\mathbf{x})} \underbrace{\left\{\ell(\mathbf{x}, \mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\mathbf{u})}\left[V^\pi(\mathbf{x}')\right]\right\}}_{Q^\pi(\mathbf{x},\mathbf{u})}, \quad \forall \mathbf{x} \in \mathcal{X}$$
- **Model-free case**: is it still possible to implement the GPI algorithm?
  - **Policy Evaluation**: given π, we saw in the previous lecture that MC or TD learning can be used to estimate V^π or Q^π
  - **Policy Improvement**: computing π' based on V^π requires access to $\ell(\mathbf{x}, \mathbf{u})$ but based on Q^π can be done **without knowing** $\ell(\mathbf{x}, \mathbf{u})$:
$$\pi'(\mathbf{x}) = \arg\min_{\mathbf{u} \in \mathcal{U}(\mathbf{x})} Q^\pi(\mathbf{x}, \mathbf{u})$$

Policy Evaluation (Recap)

- Given π, iterate $\mathcal{T}_\pi$ to compute V^π or Q^π via Dynamic Programming (DP), Temporal Difference (TD), or Monte Carlo (MC)
- DP needs a model but TD and MC are model-free
- **Value function**:

$$DP : \mathcal{T}_\pi[V](\mathbf{x}_t) = \ell(\mathbf{x}_t, \pi(\mathbf{x}_t)) + \gamma \mathbb{E}_{\mathbf{x}_{t+1} \sim p_f(\cdot | \mathbf{x}_t, \pi(\mathbf{x}_t))} \left[V(\mathbf{x}_{t+1}) \right]$$

$$TD : \mathcal{T}_\pi[V](\mathbf{x}_t) \approx V(\mathbf{x}_t) + \alpha \left[\ell(\mathbf{x}_t, \mathbf{u}_t) + \gamma V(\mathbf{x}_{t+1}) - V(\mathbf{x}_t) \right]$$

$$MC : \mathcal{T}_\pi[V](\mathbf{x}_t) \approx V(\mathbf{x}_t) + \alpha \left[\sum_{k=0}^{T-t-1} \gamma^k \ell(\mathbf{x}_{t+k}, \mathbf{u}_{t+k}) + \gamma^{T-t} \mathfrak{q}(\mathbf{x}_T) - V(\mathbf{x}_t) \right]$$

- **Q function**:

$$DP : \mathcal{T}_\pi[Q](x_t, u_t) = \ell(x_t, u_t) + \gamma \mathbb{E}_{x_{t+1} \sim p_f(\cdot | x_t, u_t)} \left[Q(x_{t+1}, \pi(x_{t+1})) \right]$$

$$TD : \mathcal{T}_\pi[Q](x_t, u_t) \approx Q(x_t, u_t) + \alpha \left[\ell(x_t, u_t) + \gamma Q(x_{t+1}, u_{t+1}) - Q(x_t, u_t) \right]$$

$$MC : \mathcal{T}_\pi[Q](x_t, u_t) \approx Q(x_t, u_t) + \alpha \left[\sum_{k=0}^{T-t-1} \gamma^k \ell(x_{t+k}, u_{t+k}) + \gamma^{T-t} \mathfrak{q}(x_T) - Q(x_t, u_t) \right]$$

Model-free Policy Improvement

- If Q^π, instead of V^π, is estimated via MC or TD, the policy improvement step can be implemented model-free, i.e., can compute $\min_{\mathbf{u}} Q^\pi(\mathbf{x}, \mathbf{u})$ without knowing the motion model p_f or the state cost ℓ
- The fact that Q^π is an approximation to the true Q-function still causes problems:
  - Picking the "best" control according to the current estimate Q^π might not be the actual best control
  - If a deterministic policy is used for Evaluation/Improvement, one will observe returns for only one of the possible controls at each state and also might not visit many states. Hence, estimating Q^π will not be possible at those never-visited states and controls.

Example: Greedy Control Selection (David Silver)

- There are two doors in front of you
- You open the left door and get reward 0
  $\ell(\mathit{left}) = 0$
- You open the right door and get reward +1
  $\ell(\mathit{right}) = -1$
- You open the right door and get reward +3
  $\ell(\mathit{right}) = -3$
- You open the right door and get reward +2
  $\ell(\mathit{right}) = -2$
- Are you sure the right door is the best long-term choice?

"Behind one door is tenure - behind the other is flipping burgers at McDonald's."

Model-free Control

- Two ideas to ensure that you do not commit to the wrong controls too early and continue exploring the state and control spaces:

  1. **Exploring Starts**: in each episode $\rho^{(k)} \sim \pi$, choose initial state-control pairs with non-zero probability among all possible pairs $\mathcal{X} \times \mathcal{U}$

  2. ϵ-**Soft Policy**: a **stochastic policy** under which every control has a non-zero probability of being chosen and hence every reachable state will have non-zero probability of being encountered

First-visit MC Policy Iteration with Exploring Starts

Algorithm 1 MC Policy Iteration with Exploring Starts

Init: $Q(\mathbf{x}, \mathbf{u}), \pi(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}$ and $\mathbf{u} \in \mathcal{U}$
loop
  Choose $(\mathbf{x}_0, \mathbf{u}_0) \in \mathcal{X} \times \mathcal{U}$ randomly ▷ exploring starts!
  Generate an episode $\rho = \mathbf{x}_0, \mathbf{u}_0, \mathbf{x}_1, \mathbf{u}_1, \ldots, \mathbf{x}_{T-1}, \mathbf{u}_{t-1}, \mathbf{x}_T$ from π
  for each $\mathbf{x}, \mathbf{u}$ in ρ **do**
    $L \leftarrow$ return following the first occurrence of $\mathbf{x}, \mathbf{u}$
    $Q(\mathbf{x}, \mathbf{u}) \leftarrow Q(\mathbf{x}, \mathbf{u}) + \alpha \left(L - Q(\mathbf{x}, \mathbf{u})\right)$
  for each $\mathbf{x}$ in ρ **do**
    $\pi(\mathbf{x}) \leftarrow \arg\min_{\mathbf{u}} Q(\mathbf{x}, \mathbf{u})$

ϵ-Greedy Exploration

- An alternative to exploring starts
- To ensure exploration it must be possible to encounter all $|\mathcal{U}(\mathbf{x})|$ controls at state $\mathbf{x}$ with non-zero probability
- ϵ-**Soft Policy**: a stochastic policy that picks each control with probability of at least $\frac{\epsilon}{|\mathcal{U}(\mathbf{x})|}$:

$$\pi(\mathbf{u}|\mathbf{x}) = \mathbb{P}(\mathbf{u}_t = \mathbf{u} \mid \mathbf{x}_t = \mathbf{x}) \geq \frac{\epsilon}{|\mathcal{U}(\mathbf{x})|} \qquad \forall \mathbf{x} \in \mathcal{X}, \mathbf{u} \in \mathcal{U}(\mathbf{x})$$

- ϵ-**Greedy Policy**: a stochastic policy that picks the best control according to $Q(\mathbf{x}, \mathbf{u})$ in the policy improvement step but ensures that all other controls are selected with a small (non-zero) probability:

$$\pi(\mathbf{u} \mid \mathbf{x}) = \mathbb{P}(\mathbf{u}_t = \mathbf{u} \mid \mathbf{x}_t = \mathbf{x}) := \begin{cases} 1 - \epsilon + \frac{\epsilon}{|\mathcal{U}(\mathbf{x})|} & \text{if } \mathbf{u} = \arg\min\limits_{\mathbf{u}' \in \mathcal{U}(\mathbf{x})} Q(\mathbf{x}, \mathbf{u}') \\ \frac{\epsilon}{|\mathcal{U}(\mathbf{x})|} & \text{otherwise} \end{cases}$$

Bellman Equations with a Stochastic Policy

- **Value function** of a stochastic policy π:

$$
\begin{aligned}
V^\pi(\mathbf{x}) :=& \mathbb{E}_{\mathbf{u}_0,\mathbf{x}_1,\mathbf{u}_1,\mathbf{x}_2,\ldots}\left[\sum_{t=0}^{\infty}\gamma^t \ell(\mathbf{x}_t,\mathbf{u}_t) \mid \mathbf{x}_0 = \mathbf{x}\right] \\
=& \mathbb{E}_{\mathbf{u}\sim\pi(\cdot|\mathbf{x})}\left[\ell(\mathbf{x},\mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}'\sim p_f(\cdot|\mathbf{x},\mathbf{u})}\left[V^\pi(\mathbf{x}')\right]\right] \\
=& \int_{\mathcal{U}(\mathbf{x})}\left[\ell(\mathbf{x},\mathbf{u}) + \gamma\int_{\mathcal{X}}\left[V^\pi(\mathbf{x}')\right] p_f(\mathbf{x}'|\mathbf{x},\mathbf{u})d\mathbf{x}'\right]\pi(\mathbf{u}|\mathbf{x})d\mathbf{u} \\
=& \mathbb{E}_{\mathbf{u}\sim\pi(\cdot|\mathbf{x})}\left[Q^\pi(\mathbf{x},\mathbf{u})\right]
\end{aligned}
$$

- **Q function** of a stochastic policy π:

$$
\begin{aligned}
Q^\pi(\mathbf{x},\mathbf{u}) :=& \ell(\mathbf{x},\mathbf{u}) + \mathbb{E}_{\mathbf{x}_1,\mathbf{u}_1,\ldots}\left[\sum_{t=1}^{\infty}\gamma^t \ell(\mathbf{x}_t,\mathbf{u}_t) \mid \mathbf{x}_0 = \mathbf{x}, \mathbf{u}_0 = \mathbf{u}\right] \\
=& \ell(\mathbf{x},\mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}'\sim p_f(\cdot|\mathbf{x},\mathbf{u}),\mathbf{u}'\sim\pi(\cdot|\mathbf{x}')}\left[Q^\pi(\mathbf{x}',\mathbf{u}')\right]
\end{aligned}
$$

ϵ-Greedy Policy Improvement

Theorem: ϵ-Greedy Policy Improvement

For any ϵ-soft policy π with associated Q^π, the ϵ-greedy policy π' with respect to Q^π is an improvement, i.e., $V^{\pi'}(\mathbf{x}) \leq V^\pi(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}$

- **Proof**:

$$\begin{aligned}
\mathbb{E}_{\mathbf{u}' \sim \pi'(\cdot|\mathbf{x})}\left[Q^\pi(\mathbf{x}, \mathbf{u}')\right] &= \sum_{\mathbf{u}' \in \mathcal{U}(\mathbf{x})} \pi'(\mathbf{u}' \mid \mathbf{x}) Q^\pi(\mathbf{x}, \mathbf{u}') \\
&= \frac{\epsilon}{|\mathcal{U}(\mathbf{x})|} \sum_{\mathbf{u}' \in \mathcal{U}(\mathbf{x})} Q^\pi(\mathbf{x}, \mathbf{u}') + (1 - \epsilon) \min_{\mathbf{u} \in \mathcal{U}(\mathbf{x})} Q^\pi(\mathbf{x}, \mathbf{u}) \\
&\leq \frac{\epsilon}{|\mathcal{U}(\mathbf{x})|} \sum_{\mathbf{u}' \in \mathcal{U}(\mathbf{x})} Q^\pi(\mathbf{x}, \mathbf{u}') + (1 - \epsilon) \sum_{\mathbf{u} \in \mathcal{U}(\mathbf{x})} \frac{\pi(\mathbf{u} \mid \mathbf{x}) - \frac{\epsilon}{|\mathcal{U}(\mathbf{x})|}}{1 - \epsilon} Q^\pi(\mathbf{x}, \mathbf{u}) \\
&= \sum_{\mathbf{u} \in \mathcal{U}(\mathbf{x})} \pi(\mathbf{u} \mid \mathbf{x}) Q^\pi(\mathbf{x}, \mathbf{u}) = V^\pi(\mathbf{x})
\end{aligned}$$

ϵ-Greedy Policy Improvement

- Then, similarity to the policy improvement theorem for deterministic policies, for all $\mathbf{x} \in \mathcal{X}$:

$$\begin{aligned}
V^{\pi}(\mathbf{x}) &\geq \mathbb{E}_{\mathbf{u}_0 \sim \pi'(\cdot|\mathbf{x})} \left[Q^{\pi}(\mathbf{x}, \mathbf{u}_0) \right] \\
&= \mathbb{E}_{\mathbf{u}_0 \sim \pi'(\cdot|\mathbf{x})} \left[\ell(\mathbf{x}, \mathbf{u}_0) + \gamma \mathbb{E}_{\mathbf{x}_1 \sim p_f(\cdot|\mathbf{x}, \mathbf{u}_0)} \left[V^{\pi}(\mathbf{x}_1) \right] \right] \\
&\geq \mathbb{E}_{\mathbf{u}_0 \sim \pi'(\cdot|\mathbf{x})} \left[\ell(\mathbf{x}, \mathbf{u}_0) + \gamma \mathbb{E}_{\mathbf{x}_1 \sim p_f(\cdot|\mathbf{x}, \mathbf{u}_0)} \left[\mathbb{E}_{\mathbf{u}_1 \sim \pi'(\cdot|\mathbf{x}_1)} \left[Q^{\pi}(\mathbf{x}_1, \mathbf{u}_1) \right] \right] \right] \\
&= \mathbb{E}_{\mathbf{u}_0 \sim \pi'(\cdot|\mathbf{x})} \left[\ell(\mathbf{x}, \mathbf{u}_0) + \gamma \mathbb{E}_{\mathbf{x}_1, \mathbf{u}_1} \left[\ell(\mathbf{x}_1, \mathbf{u}_1) + \gamma \mathbb{E}_{\mathbf{x}_2} V^{\pi}(\mathbf{x}_2) \right] \right] \\
&\geq \cdots \geq \mathbb{E}_{\rho_0 \sim \pi'} \left[\sum_{t=0}^{\infty} \gamma^t \ell(\mathbf{x}_t, \mathbf{u}_t) \middle| \mathbf{x}_0 = \mathbf{x} \right] = V^{\pi'}(\mathbf{x})
\end{aligned}$$

First-visit MC Policy Iteration with ϵ-Greedy Improvement

Algorithm 2 First-visit MC Policy Iteration with ϵ-Greedy Improvement

Init: $Q(\mathbf{x}, \mathbf{u})$, $\pi(\mathbf{u}|\mathbf{x})$ (ϵ-soft policy) for all $\mathbf{x} \in \mathcal{X}$ and $\mathbf{u} \in \mathcal{U}$
loop
  Generate an episode $\rho := \mathbf{x}_0, \mathbf{u}_0, \mathbf{x}_1, \mathbf{u}_1, \ldots, \mathbf{x}_{T-1}, \mathbf{u}_{t-1}, \mathbf{x}_T$ from π
  for each $\mathbf{x}, \mathbf{u}$ in ρ **do**
    $L \leftarrow$ return following the first occurrence of $\mathbf{x}, \mathbf{u}$
    $Q(\mathbf{x}, \mathbf{u}) \leftarrow Q(\mathbf{x}, \mathbf{u}) + \alpha \left(L - Q(\mathbf{x}, \mathbf{u})\right)$
  for each $\mathbf{x}$ in ρ **do**
    $\mathbf{u}^* \leftarrow \arg\min_{\mathbf{u}} Q(\mathbf{x}, \mathbf{u})$
    $\pi(\mathbf{u}|\mathbf{x}) \leftarrow \begin{cases} 1 - \epsilon + \frac{\epsilon}{|\mathcal{U}(\mathbf{x})|} & \text{if } \mathbf{u} = \mathbf{u}^* \\ \frac{\epsilon}{|\mathcal{U}(\mathbf{x})|} & \text{if } \mathbf{u} \neq \mathbf{u}^* \end{cases}$

Temporal-Difference Control

- TD prediction has several advantages over MC prediction:
  - Works with incomplete episodes
  - Can perform online updates to Q^π after every transition
  - The TD estimate of Q^π has lower variance than the MC one
- TD in the policy iteration algorithm:
  - Use TD for policy evaluation
  - Can update $Q(\mathbf{x}, \mathbf{u})$ after every transition within an episode
  - Use an ϵ-greedy policy for policy improvement because we still need to trade off exploration and exploitation

TD Policy Iteration with ϵ-Greedy Improvement (SARSA)

- **SARSA**: estimates the action-value function Q^π using TD updates after every $S_t, A_t, R_{t+1}, S_{t+1}, A_{t+1}$ transition:

$$Q(\mathbf{x}_t, \mathbf{u}_t) \leftarrow Q(\mathbf{x}_t, \mathbf{u}_t) + \alpha \left[\ell(\mathbf{x}_t, \mathbf{u}_t) + \gamma Q(\mathbf{x}_{t+1}, \mathbf{u}_{t+1}) - Q(\mathbf{x}_t, \mathbf{u}_t)\right]$$

- Ensures exploration via an ϵ-greedy policy in the policy improvement step

Algorithm 3 SARSA

Init: $Q(\mathbf{x}, \mathbf{u})$ for all $\mathbf{x} \in \mathcal{X}$ and all $\mathbf{u} \in \mathcal{U}$
loop
    $\pi \leftarrow \epsilon$-greedy policy derived from Q
    Generate episode $\rho := \mathbf{x}_0, \mathbf{u}_0, \mathbf{x}_1, \mathbf{u}_1, \ldots, \mathbf{x}_{T-1}, \mathbf{u}_{t-1}, \mathbf{x}_T$ from π
    for $(\mathbf{x}, \mathbf{u}, \mathbf{x}', \mathbf{u}') \in \rho$ **do**
        $Q(\mathbf{x}, \mathbf{u}) \leftarrow Q(\mathbf{x}, \mathbf{u}) + \alpha \left[\ell(\mathbf{x}, \mathbf{u}) + \gamma Q(\mathbf{x}', \mathbf{u}') - Q(\mathbf{x}, \mathbf{u})\right]$

Convergence of Model-free Policy Iteration

- **Greedy in the Limit with Infinite Exploration** (GLIE):
  - All state-control pairs are explored infinitely many times: $\lim_{k\to\infty} N_k(\mathbf{x},\mathbf{u}) = \infty$
  - The ϵ-greedy policy converges to a greedy policy wrt $\mathbf{u}^* = \arg\min_{\mathbf{u}\in\mathcal{U}(\mathbf{x})} Q(\mathbf{x},\mathbf{u})$.
- Example: If $\epsilon_k = \frac{1}{k}$, then ϵ-greedy is GLIE

$$\pi_k(\mathbf{u} \mid \mathbf{x}) := \begin{cases} 1 - \epsilon_k + \frac{\epsilon_k}{|\mathcal{U}(\mathbf{x})|} & \text{if } \mathbf{u} = \mathbf{u}^* \\ \frac{\epsilon_k}{|\mathcal{U}(\mathbf{x})|} & \text{if } \mathbf{u} \neq \mathbf{u}^* \end{cases} \qquad \lim_{k\to\infty} \pi_k(\mathbf{u} \mid \mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{u} = \mathbf{u}^* \\ 0 & \text{if } \mathbf{u} \neq \mathbf{u}^* \end{cases}$$

Theorem: Convergence of Model-free Policy Iteration

Both MC Policy Iteration and SARSA converge to the optimal action-value function, $Q(\mathbf{x},\mathbf{u}) \to Q^*(\mathbf{x},\mathbf{u})$, as the number of episodes $k \to \infty$ as long as:

- the sequence of ϵ-greedy policies $\pi_k(\mathbf{u} \mid \mathbf{x})$ is GLIE,
- the sequence of step sizes α_k is Robbins-Monro.

On-Policy vs Off-Policy Learning

- **On-policy Prediction**: estimate V^π or Q^π using experience from π
- **Off-policy Prediction**: estimate V^π or Q^π using experience from μ
- On-policy methods:
  - evaluate or improve the policy π that is used to make decisions and collect experience
  - require well-designed exploration functions
  - empirically successful with function approximation
- Off-policy methods:
  - evaluate or improve a policy π that is different from the (behavior) policy μ used to generate data
  - can use an effective exploratory policy μ to generate data while learning about an optimal policy
  - can learn from observing other agents (or humans)
  - can re-use experience from old policies $\pi_1, \pi_2, \ldots, \pi_{k-1}$
  - can learn about multiple policies while following one policy
  - have problems with function approximation and eligibility traces

Importance Sampling for Off-policy Learning

- Off-policy learning: use returns generated from μ to evaluate π
- The stage costs obtained from μ, need to be re-weighted according to the similarity (i.e., likelihood) of the states encountered by π
- **Importance Sampling**: estimates the expectation of a function $\ell(x)$ with respect to a probability density function $p(x)$ by computing a re-weighted expectation over a different probability density $q(x)$:

$$\begin{aligned}\mathbb{E}_{x\sim p(\cdot)}[\ell(x)] &= \int p(x)\ell(x)dx \\ &= \int q(x)\frac{p(x)}{q(x)}\ell(x)dx = \mathbb{E}_{x\sim q(\cdot)}\left[\frac{p(x)}{q(x)}\ell(x)\right]\end{aligned}$$

Importance Sampling for Off-policy MC Learning

- To use returns generated from μ to evaluate π via MC, weight the long-term cost L_t via importance-sampling corrections along the whole episode:

$$L_t^{\pi/\mu} = \frac{\pi(\mathbf{u}_t|\mathbf{x}_t)}{\mu(\mathbf{u}_t|\mathbf{x}_t)} \frac{\pi(\mathbf{u}_{t+1}|\mathbf{x}_{t+1})}{\mu(\mathbf{u}_{t+1}|\mathbf{x}_{t+1})} \cdots \frac{\pi(\mathbf{u}_{T-1}|\mathbf{x}_{T-1})}{\mu(\mathbf{u}_{T-1}|\mathbf{x}_{T-1})} L_t$$

- Update the value estimate towards the *corrected return*:

$$V^\pi(\mathbf{x}_t) \leftarrow V^\pi(\mathbf{x}_t) + \alpha \left(L_t^{\pi/\mu} - V^\pi(\mathbf{x}_t) \right)$$

- **Note**: importance sampling in MC can dramatically increase variance

Importance Sampling for Off-policy TD Learning

- To use returns generated from μ to evaluate π via TD, weight the TD target $\ell(\mathbf{x}, \mathbf{u}) + \gamma V(\mathbf{x}')$ by importance sampling:

$$V^\pi(\mathbf{x}_t) \leftarrow V^\pi(\mathbf{x}_t) + \alpha \left(\frac{\pi(\mathbf{u}_t \mid \mathbf{x}_t)}{\mu(\mathbf{u}_t \mid \mathbf{x}_t)} \left(\ell(\mathbf{x}_t, \mathbf{u}_t) + \gamma V^\pi(\mathbf{x}_{t+1})\right) - V^\pi(\mathbf{x}_t) \right)$$

- Importance sampling in TD is much lower variance than in MC and the policies need to be similar (i.e., μ should not be zero when π is non-zero) over a single step only

Off-policy TD Control without Importance Sampling

- **Q-Learning** (Watkins, 1989): one of the early breakthroughs in reinforcement learning was the development of an off-policy TD algorithm that does not use importance sampling
- Q-Learning approximates $\mathcal{T}_*[Q](\mathbf{x}, \mathbf{u})$ directly using samples:

$$Q(\mathbf{x}_t, \mathbf{u}_t) \leftarrow Q(\mathbf{x}_t, \mathbf{u}_t) + \alpha \left[\ell(\mathbf{x}_t, \mathbf{u}_t) + \gamma \min_{\mathbf{u} \in \mathcal{U}(\mathbf{x}_{t+1})} Q(\mathbf{x}_{t+1}, \mathbf{u}) - Q(\mathbf{x}_t, \mathbf{u}_t) \right]$$

- The learned Q function eventually approximates Q^* **regardless of the policy being followed**!

Theorem: Convergence of Q-Learning

Q-Learning converges almost surely to Q^* assuming all state-control pairs continue to be updated and the sequence of step sizes α_k is Robbins-Monro.

- C. J. Watkins and P. Dayan. "Q-learning," Machine learning, 1992.

Q-Learning: Off-policy TD Learning

Algorithm 4 Q-Learning

```
Init: Q(x, u) for all x ∈ 𝒳 and all u ∈ 𝒰
loop
    π ← ε-greedy policy derived from Q                ▷ π can be arbitrary!
    Generate episode ρ := x_0, u_0, x_1, u_1, ..., x_{T−1}, u_{t−1}, x_T from π
    for (x, u, x') ∈ ρ do
        Q(x, u) ← Q(x, u) + α [ℓ(x, u) + γ min_{u'} Q(x', u') − Q(x, u)]
```

Relationship Between Full and Sample Backups

Full Backups (DP)	**Sample Backups** (TD)
Policy Evaluation $V(\mathbf{x}) \leftarrow \mathcal{T}_\pi[V](\mathbf{x}) = \ell(\mathbf{x}, \pi(\mathbf{x})) + \gamma \mathbb{E}_{\mathbf{x}'}\left[V(\mathbf{x}')\right]$	**TD Prediction** $V(\mathbf{x}) \leftarrow V(\mathbf{x}) + \alpha(\ell(\mathbf{x}, \mathbf{u}) + \gamma V(\mathbf{x}') - V(\mathbf{x}))$
Policy Q-Evaluation $Q(\mathbf{x}, \mathbf{u}) \leftarrow \mathcal{T}_\pi[Q](\mathbf{x}, \mathbf{u}) = \ell(\mathbf{x}, \mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}'}\left[Q(\mathbf{x}', \pi(\mathbf{x}'))\right]$	**TD Prediction Step in SARSA** $Q(\mathbf{x}, \mathbf{u}) \leftarrow Q(\mathbf{x}, \mathbf{u}) + \alpha(\ell(\mathbf{x}, \mathbf{u}) + \gamma Q(\mathbf{x}', \mathbf{u}') - Q(\mathbf{x}, \mathbf{u}))$
Value Iteration $V(\mathbf{x}) \leftarrow \mathcal{T}_*[V](\mathbf{x}) = \min_{\mathbf{u}} \left\{\ell(\mathbf{x}, \mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}'}\left[V(\mathbf{x}')\right]\right\}$	**N/A**
Q-Value Iteration $Q(\mathbf{x}, \mathbf{u}) \leftarrow \mathcal{T}_*[Q](\mathbf{x}, \mathbf{u}) = \ell(\mathbf{x}, \mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}'}\left[\min_{\mathbf{u}'} Q(\mathbf{x}', \mathbf{u}')\right]$	**Q-Learning** $Q(\mathbf{x}, \mathbf{u}) \leftarrow Q(\mathbf{x}, \mathbf{u}) + \alpha\left(\ell(\mathbf{x}, \mathbf{u}) + \gamma \min_{\mathbf{u}'} Q(\mathbf{x}', \mathbf{u}') - Q(\mathbf{x}, \mathbf{u})\right)$

Batch Sampling-based Q-Value Iteration

Algorithm 5 Batch Sampling-based Q-Value Iteration

1: **Init**: $Q_0(\mathbf{x}, \mathbf{u})$ for all $\mathbf{x} \in \mathcal{X}$ and all $\mathbf{u} \in \mathcal{U}$
2: **loop**
3: $\pi \leftarrow \epsilon$-greedy policy derived from Q_i
4: Generate episodes $\{\rho^{(k)}\}_{k=1}^{K}$ from π
5: **for** $(\mathbf{x}, \mathbf{u}) \in \mathcal{X} \times \mathcal{U}$ **do**
6: $$Q_{i+1}(\mathbf{x}, \mathbf{u}) = \frac{1}{K} \sum_{k=1}^{K} \frac{\sum_{t=0}^{T^{(k)}} \mathcal{T}_*[Q_i](\mathbf{x}_t^{(k)}, \mathbf{u}_t^{(k)}, \mathbf{x}_{t+1}^{(k)}) \mathbb{1}\{(\mathbf{x}_t^{(k)}, \mathbf{u}_t^{(k)}) = (\mathbf{x}, \mathbf{u})\}}{\sum_{t=0}^{T^{(k)}} \mathbb{1}\{(\mathbf{x}_t^{(k)}, \mathbf{u}_t^{(k)}) = (\mathbf{x}, \mathbf{u})\}}$$

- Batch Sampling-based Q-Value Iteration behaves like $Q_{i+1} = \mathcal{T}_*[Q_i] + \text{noise}$. Does it actually converge?

Least-squares Backup Version

- $Q_{i+1}(\mathbf{x}, \mathbf{u}) = \textbf{mean}\left\{\mathcal{T}_*[Q_i](\mathbf{x}_t^{(k)}, \mathbf{u}_t^{(k)}, \mathbf{x}_{t+1}^{(k)}),\ \forall k, t \text{ such that } (\mathbf{x}_t^{(k)}, \mathbf{u}_t^{(k)}) = (\mathbf{x}, \mathbf{u})\right\}$
- Note that: $\textbf{mean}\left\{\mathbf{x}^{(k)}\right\} = \arg\min_{\mathbf{x}} \sum_{k=1}^{K} \|\mathbf{x}^{(k)} - \mathbf{x}\|^2$
- $$Q_{i+1}(\mathbf{x}, \mathbf{u}) = \arg\min_{q} \sum_{k=1}^{K} \sum_{(\mathbf{x}_t^{(k)}, \mathbf{u}_t^{(k)}) = (\mathbf{x}, \mathbf{u})} \left\|\mathcal{T}_*[Q_i](\mathbf{x}_t^{(k)}, \mathbf{u}_t^{(k)}, \mathbf{x}_{t+1}^{(k)}) - q\right\|^2$$
- $$Q_{i+1} = \arg\min_{Q} \sum_{k=1}^{K} \sum_{t=0}^{T^{(k)}} \left\|\mathcal{T}_*[Q_i](\mathbf{x}_t^{(k)}, \mathbf{u}_t^{(k)}, \mathbf{x}_{t+1}^{(k)}) - Q(\mathbf{x}_t^{(k)}, \mathbf{u}_t^{(k)})\right\|^2$$

Algorithm 6 Batch Least-squares Q-Value Iteration

1: **Init**: $Q_0(\mathbf{x}, \mathbf{u})$ for all $\mathbf{x} \in \mathcal{X}$ and all $\mathbf{u} \in \mathcal{U}$
2: **loop**
3: $\pi \leftarrow \epsilon$-greedy policy derived from Q_i
4: Generate episodes $\{\rho^{(k)}\}_{k=1}^{K}$ from π
5: $$Q_{i+1} = \arg\min_{Q} \sum_{k=1}^{K} \sum_{t=0}^{T^{(k)}} \left\|\mathcal{T}_*[Q_i](\mathbf{x}_t^{(k)}, \mathbf{u}_t^{(k)}, \mathbf{x}_{t+1}^{(k)}) - Q(\mathbf{x}_t^{(k)}, \mathbf{u}_t^{(k)})\right\|^2$$

Small Steps in the Backup Direction

- Full backup: $Q_{i+1} \leftarrow \mathcal{T}_*[Q_i] + \text{noise}$
- Partial backup: $Q_{i+1} \leftarrow \alpha \mathcal{T}_*[Q_i] + (1-\alpha) Q_i + \text{noise}$
- Equivalent to a gradient step on squared error objective function:

$$
\begin{aligned}
Q_{i+1} &\leftarrow \alpha \mathcal{T}_*[Q_i] + (1-\alpha) Q_i + \text{noise} \\
&= Q_i + \alpha \left(\mathcal{T}_*[Q_i] - Q_i \right) + \text{noise} \\
&= Q_i - \alpha \left(\frac{1}{2} \nabla_Q \| \mathcal{T}_*[Q_i] - Q \|^2 \Big|_{Q=Q_i} + \text{noise} \right)
\end{aligned}
$$

- Behaves like stochastic gradient descent for $f(Q) := \frac{1}{2} \| \mathcal{T}_*[Q_i] - Q \|^2$ but the objective is changing, i.e., $\mathcal{T}_*[Q_i]$ is a moving target
- **Stochastic Approximation Theory**: a "partial update" to ensure contraction + appropriate step size α implies convergence to the contraction fixed point: $\lim_{i \to \infty} Q_i = Q^*$
- T. Jaakkola, M. Jordan, S. Singh, "On the convergence of stochastic iterative dynamic programming algorithms," Neural computation, 1994

Least-squares Partial Backup Version

Algorithm 7 Batch Gradient Least-squares Q-Value Iteration

1: **Init**: $Q_0(\mathbf{x}, \mathbf{u})$ for all $\mathbf{x} \in \mathcal{X}$ and all $\mathbf{u} \in \mathcal{U}$
2: **loop**
3: $\pi \leftarrow \epsilon$-greedy policy derived from Q_i
4: Generate episodes $\{\rho^{(k)}\}_{k=1}^{K}$ from π
5: $Q_{i+1} \leftarrow Q_i - \frac{\alpha}{2} \nabla_Q \left[\sum_{k=1}^{K} \sum_{t=0}^{T^{(k)}} \| \mathcal{T}_*[Q_i](\mathbf{x}_t^{(k)}, \mathbf{u}_t^{(k)}, \mathbf{x}_{t+1}^{(k)}) - Q(\mathbf{x}_t^{(k)}, \mathbf{u}_t^{(k)}) \|^2 \right] \Big|_{Q=Q_i}$

- Watkins Q-learning is a special case with $T = 1$