

ECE276B: Planning & Learning in Robotics

Lecture 2: Markov Chains

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Definition of Markov Chain

▶ Intuitive definition:

- ▶ In a Markov Chain, the distribution of the state $x_{t+1} \mid x_{0:t}$ depends only on x_t and not the rest of the history $x_{0:t-1}$ (a memoryless stochastic process)
- ▶ “The future is independent of the past given the present” (**Markov Assumption**)
- ▶ The state x_t captures all information about the history, i.e., once the state is known, the history may be thrown away

▶ Formal definition:

- ▶ A **stochastic process** is an indexed collection of random variables $\{x_0, x_1, \dots\}$ whose range is a measurable space $(\mathcal{X}, \mathcal{F})$
 - ▶ example: time series of weekly demands for a product
- ▶ A temporally homogeneous **Markov chain** is a stochastic process $\{x_0, x_1, \dots\}$ such that:
 - ▶ $x_0 \sim p_0(\cdot)$ for a prior pdf $p_0(\cdot)$ on $(\mathcal{X}, \mathcal{F})$
 - ▶ $x_{t+1} \sim p_f(\cdot \mid x_t)$ for a conditional pdf $p_f(\cdot \mid x_t)$ on $(\mathcal{X}, \mathcal{F})$

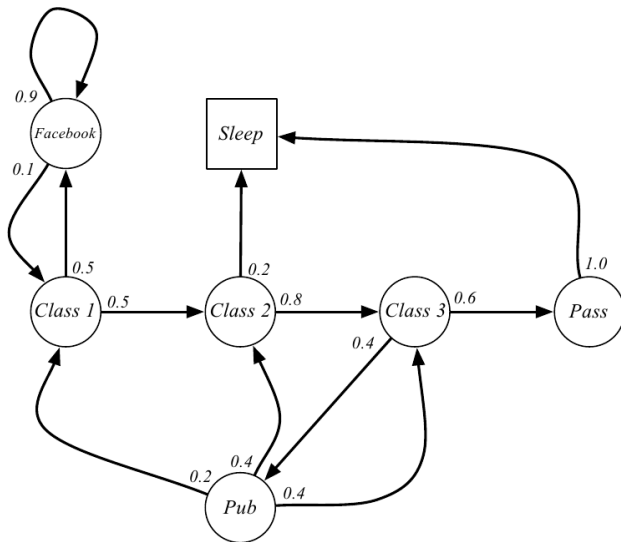
Markov Chain

A **Markov Chain** is a stochastic process defined by a tuple $(\mathcal{X}, p_0, p_f, T)$:

- ▶ \mathcal{X} is a discrete/continuous set of states
 - ▶ p_0 is a prior pmf/pdf defined on \mathcal{X}
 - ▶ $p_f(\cdot | \mathbf{x})$ is a conditional pmf/pdf defined on \mathcal{X} for given $\mathbf{x} \in \mathcal{X}$ that specifies the stochastic process transitions.
 - ▶ T is a finite/infinite time horizon
- ▶ When there is a finite number of states, $\mathcal{X} := \{1, \dots, N\}$, the motion model p_f is a probability mass function (pmf) and can be summarized by an $N \times N$ matrix with elements:

$$P_{ij} := \mathbb{P}(x_{t+1} = j | x_t = i) = p_f(j | x_t = i)$$

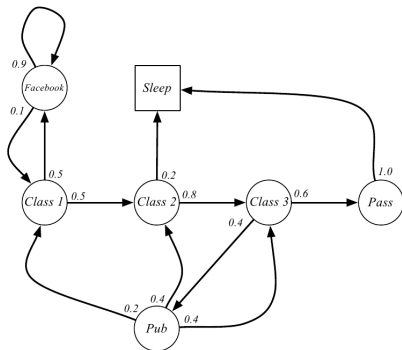
Example: Student Markov Chain



Example: Student Markov Chain

▶ Sample paths:

- ▶ C1 C2 C3 Pass Sleep
- ▶ C1 FB FB C1 C2 Sleep
- ▶ C1 C2 C3 Pub C2 C3 Pass Sleep
- ▶ C1 FB FB C1 C2 C3 Pub C1 FB
FB FB C1 C2 Sleep



▶ Transition matrix:

$$P = \begin{matrix} FB \\ C1 \\ C2 \\ C3 \\ Pub \\ Pass \\ Sleep \end{matrix} \begin{bmatrix} 0.9 & 0.1 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0 & 0.4 & 0.6 & 0 \\ 0 & 0.2 & 0.4 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Chapman-Kolmogorov Equation

- ▶ **n -step transition probabilities** of a time-homogeneous Markov chain on $\mathcal{X} = \{1, \dots, N\}$

$$P_{ij}^{(n)} := \mathbb{P}(x_{t+n} = j \mid x_t = i) = \mathbb{P}(x_n = j \mid x_0 = i)$$

- ▶ **Chapman-Kolmogorov:** the n -step transition probabilities can be obtained recursively from the 1-step transition probabilities:

$$P_{ij}^{(n)} = \sum_{k=1}^N P_{ik}^{(m)} P_{kj}^{(n-m)}, \quad \forall i, j, n, 0 \leq m \leq n$$

$$P^{(n)} = \underbrace{P \dots P}_{n \text{ times}} = P^n$$

- ▶ Given the transition matrix P and a vector $\mathbf{p}_0 := [p_0(1), \dots, p_0(N)]^\top$ of prior probabilities, the vector of probabilities \mathbf{p}_t after t steps is:

$$\mathbf{p}_t^\top = \mathbf{p}_0^\top P^t$$

Example: Student Markov Chain

$$P = \begin{matrix} FB \\ C1 \\ C2 \\ C3 \\ Pub \\ Pass \\ Sleep \end{matrix} \begin{bmatrix} 0.9 & 0.1 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0 & 0.4 & 0.6 & 0 \\ 0 & 0.2 & 0.4 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^2 = \begin{matrix} FB \\ C1 \\ C2 \\ C3 \\ Pub \\ Pass \\ Sleep \end{matrix} \begin{bmatrix} 0.86 & 0.09 & 0.05 & 0 & 0 & 0 & 0 \\ 0.45 & 0.05 & 0 & 0.4 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 & 0.32 & 0.48 & 0.2 \\ 0 & 0.08 & 0.16 & 0.16 & 0 & 0 & 0.6 \\ 0.1 & 0 & 0.1 & 0.32 & 0.16 & 0.24 & 0.08 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^{100} = \begin{matrix} FB \\ C1 \\ C2 \\ C3 \\ Pub \\ Pass \\ Sleep \end{matrix} \begin{bmatrix} 0.01 & 0 & 0 & 0 & 0 & 0 & 0.99 \\ 0.01 & 0 & 0 & 0 & 0 & 0 & 0.99 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

First Passage Time

- ▶ **First Passage Time:** the number of transitions necessary to go from $x_0 = i$ to state j for the first time is a random variable

$$\tau_{ij} := \inf\{t \geq 1 \mid x_t = j, x_0 = i\}$$

- ▶ **Recurrence Time:** the first passage time τ_{ii} to go from $x_0 = i$ to $j = i$
- ▶ **Probability of first passage in n steps:** $\rho_{ij}^{(n)} := \mathbb{P}(\tau_j = n \mid x_0 = i)$

$$\rho_{ij}^{(1)} = P_{ij}$$

$$\rho_{ij}^{(2)} = [P^2]_{ij} - \rho_{ij}^{(1)} P_{jj} \quad (\text{first time we visit } j \text{ should not be } 1!)$$

\vdots

$$\rho_{ij}^{(n)} = [P^n]_{ij} - \rho_{ij}^{(1)} [P^{n-1}]_{jj} - \rho_{ij}^{(2)} [P^{n-2}]_{jj} - \dots - \rho_{ij}^{(n-1)} P_{jj}$$

- ▶ **Probability of first passage:** $\rho_{ij} := \mathbb{P}(\tau_j < \infty \mid x_0 = i) = \sum_{n=1}^{\infty} \rho_{ij}^{(n)}$
- ▶ **Number of visits to j up to time n :**

$$v_j^{(n)} := \sum_{t=0}^n \mathbb{1}\{x_t = j\} \quad v_j := \lim_{n \rightarrow \infty} v_j^{(n)}$$

Recurrence and Transience

- ▶ **Absorbing state:** a state j such that $P_{jj} = 1$
- ▶ **Transient state:** a state j such that $\rho_{jj} < 1$
- ▶ **Recurrent state:** a state j such that $\rho_{jj} = 1$
- ▶ **Positive recurrent state:** a recurrent state j with $\mathbb{E}[\tau_j \mid x_0 = j] < \infty$
- ▶ **Null recurrent state:** a recurrent state j with $\mathbb{E}[\tau_j \mid x_0 = j] = \infty$
- ▶ **Periodic state:** can only be visited at integer multiples of t
- ▶ **Ergodic state:** a positive recurrent state that is aperiodic

Recurrence and Transience

Total Number of Visits Lemma

$$\mathbb{P}(v_j \geq k + 1 \mid x_0 = j) = \rho_{jj}^k \text{ for all } k \geq 0$$

Proof: By the (strong) Markov property and induction ($\mathbb{P}(v_j \geq k + 1 \mid x_0 = j) = \rho_{jj} \mathbb{P}(v_j \geq k \mid x_0 = j)$).

0–1 Law for the Total Number of Visits

$$j \text{ is recurrent iff } \mathbb{E}[v_j \mid x_0 = j] = \infty$$

Proof: Since v_j is discrete, we can write $v_j = \sum_{k=0}^{\infty} \mathbb{1}\{v_j > k\}$ and

$$\mathbb{E}[v_j \mid x_0 = j] = \sum_{k=0}^{\infty} \mathbb{P}(v_j \geq k + 1 \mid x_0 = j) = \sum_{k=0}^{\infty} \rho_{jj}^k = \frac{\rho_{jj}}{1 - \rho_{jj}}$$

Theorem: Recurrence is contagious

$$i \text{ is recurrent and } \rho_{ij} > 0 \quad \Rightarrow \quad j \text{ is recurrent and } \rho_{ji} = 1$$

Classification of Markov Chains

- ▶ **Absorbing Markov Chain:** contains at least one absorbing state that can be reached from every other state (not necessarily in one step)
- ▶ **Irreducible Markov Chain:** it is possible to go from every state to every state (not necessarily in one step)
- ▶ **Ergodic Markov Chain:** an aperiodic, irreducible and positive recurrent Markov chain
- ▶ **Stationary distribution:** a vector $\mathbf{w} \in \{\mathbf{p} \in [0, 1]^N \mid \mathbf{1}^\top \mathbf{p} = 1\}$ such that $\mathbf{w}^\top P = \mathbf{w}^\top$
 - ▶ Absorbing chains have stationary distributions with nonzero elements only in absorbing states
 - ▶ Ergodic chains have a unique stationary distribution (Perron-Frobenius Theorem)
 - ▶ Some periodic chains only satisfy a weaker condition, where $w_j > 0$ only for recurrent states and w_j is the frequency $\frac{v_j^{(n)}}{n+1}$ of being in state j as $n \rightarrow \infty$

Absorbing Markov Chains

► Interesting questions:

Q1: On average, how many times is the process in state j ?

Q2: What is the probability that the state will eventually be absorbed?

Q3: What is the expected absorption time?

Q4: What is the probability of being absorbed by j given that we started in i ?

Absorbing Markov Chains

- ▶ **Canonical form:** reorder the states so that the transient ones come first: $P = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}$

- ▶ One can show that $P^n = \begin{bmatrix} Q^n & * \\ 0 & I \end{bmatrix}$ and $Q^n \rightarrow 0$ as $n \rightarrow \infty$

Proof: If j is transient, then $\rho_{ij} < 1$ and from the 0-1 Law:

$$\infty > \mathbb{E}[v_j \mid x_0 = i] = \mathbb{E}\left[\sum_{n=0}^{\infty} \mathbb{1}\{x_n = j\} \mid x_0 = i\right] = \sum_{n=0}^{\infty} [P^n]_{ij}$$

- ▶ **Fundamental matrix:** $Z^A = (I - Q)^{-1} = \sum_{n=0}^{\infty} Q^n$ exists for an absorbing Markov chain
 - ▶ Expected number of times the chain is in state j : $Z_{ij}^A = \mathbb{E}[v_j \mid x_0 = i]$
 - ▶ Expected absorption time when starting from state i : $\sum_j Z_{ij}^A$
 - ▶ Let $B = Z^A R$. The probability of reaching absorbing state j starting from state i is B_{ij}

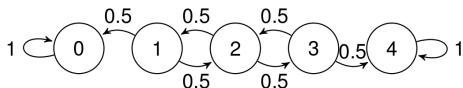
Example: Drunkard's Walk

- ▶ Transition matrix:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- ▶ Canonical form:

$$P = \begin{bmatrix} 0 & 0.5 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$



- ▶ Fundamental matrix:

$$Z^A = (I - Q)^{-1} = \begin{bmatrix} 1.5 & 1 & 0.5 \\ 1 & 2 & 1 \\ 0.5 & 1 & 1.5 \end{bmatrix}$$

Perron-Frobenius Theorem

Theorem

Let P be the transition matrix of an irreducible, aperiodic, finite, time-homogeneous Markov chain with stationary distribution \mathbf{w} . Then

- ▶ 1 is the eigenvalue of max modulus, i.e., $|\lambda| < 1$ for all other eigenvalues
- ▶ 1 is a simple eigenvalue, i.e., the associated eigenspace and left-eigenspace have dimension 1
- ▶ The eigenvector associated with 1 is $\mathbf{1}$
- ▶ The unique left eigenvector \mathbf{w} is nonnegative and

$$\lim_{n \rightarrow \infty} P^n = \mathbf{1}\mathbf{w}^\top$$

Hence, \mathbf{w} is the unique stationary distribution for the Markov chain and any initial distribution converges to it.

Fundamental Matrix for Ergodic Chains

- ▶ We can try to get a fundamental matrix as in the absorbing case but $(I - P)^{-1}$ does not exist because $P\mathbf{1} = \mathbf{1}$ (Perron-Frobenius)
- ▶ $I + Q + Q^2 + \dots = (I - Q)^{-1}$ converges because $Q^n \rightarrow 0$
- ▶ Try $I + (P - \mathbf{1}\mathbf{w}^\top) + (P^2 - \mathbf{1}\mathbf{w}^\top) + \dots$ because $P^n \rightarrow \mathbf{1}\mathbf{w}^\top$ (Perron-Frobenius)
- ▶ Note that $P\mathbf{1}\mathbf{w}^\top = \mathbf{1}\mathbf{w}^\top$ and $(\mathbf{1}\mathbf{w}^\top)^2 = \mathbf{1}\mathbf{w}^\top\mathbf{1}\mathbf{w}^\top = \mathbf{1}\mathbf{w}^\top$

$$\begin{aligned}(P - \mathbf{1}\mathbf{w}^\top)^n &= \sum_{i=0}^n (-1)^i \binom{n}{i} P^{n-i} (\mathbf{1}\mathbf{w}^\top)^i = P^n + \sum_{i=1}^n (-1)^i \binom{n}{i} (\mathbf{1}\mathbf{w}^\top)^i \\ &= P^n + \underbrace{\left[\sum_{i=1}^n (-1)^i \binom{n}{i} \right]}_{(1-1)^{n-1}} (\mathbf{1}\mathbf{w}^\top) = P^n - \mathbf{1}\mathbf{w}^\top\end{aligned}$$

- ▶ Thus, the following inverse exists:

$$I + \sum_{n=1}^{\infty} (P^n - \mathbf{1}\mathbf{w}^\top) = I + \sum_{n=1}^{\infty} (P - \mathbf{1}\mathbf{w}^\top)^n = (I - P + \mathbf{1}\mathbf{w}^\top)^{-1}$$

Fundamental Matrix for Ergodic Chains

- ▶ **Fundamental matrix:** $Z^E := (I - P + \mathbf{1}\mathbf{w}^\top)^{-1}$ where P is the transition matrix and \mathbf{w} is the stationary distribution.
- ▶ **Properties:** $\mathbf{w}^\top Z^E = \mathbf{w}^\top$, $Z^E \mathbf{1} = \mathbf{1}$, and $Z^E(I - P) = I - \mathbf{1}\mathbf{w}^\top$
- ▶ **Mean first passage time:** $m_{ij} := \mathbb{E}[\tau_j \mid x_0 = i] = \frac{Z_{jj}^E - Z_{ij}^E}{w_j}$

Example: Land of Oz

- ▶ Transition matrix:

$$P = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.5 & 0 & 0.5 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

- ▶ Stationary distribution:

$$\mathbf{w}^\top = [0.4 \quad 0.2 \quad 0.4]$$

- ▶ Fundamental matrix:

$$I - P + \mathbf{1}\mathbf{w}^\top = \begin{bmatrix} 0.9 & -0.05 & 0.15 \\ -0.1 & 1.2 & -0.1 \\ 0.15 & -0.05 & 0.9 \end{bmatrix}$$

$$Z^E = \begin{bmatrix} 1.147 & 0.04 & -0.187 \\ 0.08 & 0.84 & 0.08 \\ -0.187 & 0.04 & 1.147 \end{bmatrix}$$

- ▶ Mean first passage time:

$$m_{12} = \frac{Z_{22}^E - Z_{12}^E}{w_2} = \frac{0.84 - 0.04}{0.2} = 4$$

