ECE276B: Planning & Learning in Robotics
Lecture 10: Infinite-Horizon Optimal Control

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Outline

Infinite-Horizon Optimal Control

Bellman Equations

Policy Evaluation

Value Iteration

Policy Iteration

Linear Programming

Finite-Horizon Stochastic Optimal Control

- Recall the finite-horizon stochastic optimal control problem:

$$\begin{aligned} \min_{\pi_{\tau:T-1}} \; & V_\tau^\pi(\mathbf{x}_\tau) := \mathbb{E}_{\mathbf{x}_{\tau+1:T}}\left[\gamma^{T-\tau}\mathfrak{q}(\mathbf{x}_T) + \sum_{t=\tau}^{T-1}\gamma^{t-\tau}\ell(\mathbf{x}_t, \pi_t(\mathbf{x}_t)) \;\middle|\; \mathbf{x}_\tau\right] \\ \text{s.t. } \; & \mathbf{x}_{t+1} \sim p_f(\cdot \mid \mathbf{x}_t, \pi_t(\mathbf{x}_t)), \qquad t = \tau, \ldots, T-1 \\ & \mathbf{x}_t \in \mathcal{X}, \;\; \pi_t(\mathbf{x}_t) \in \mathcal{U} \end{aligned}$$

$\mathbf{x} \in \mathcal{X}$	state
$\mathbf{u} \in \mathcal{U}$	control
$p_f(\mathbf{x}' \mid \mathbf{x}, \mathbf{u})$	motion model
$\mathbf{x}' = f(\mathbf{x}, \mathbf{u}, \mathbf{w})$	motion model
$\ell(\mathbf{x}, \mathbf{u})$	stage cost
$\mathfrak{q}(\mathbf{x})$	terminal cost
$T \in \mathbb{N}$	planning horizon
$\gamma \in [0, 1]$	discount factor
$\pi_t(\mathbf{x})$	policy function at time t
$V_t^\pi(\mathbf{x})$	value function at state $\mathbf{x}$, time t, under policy $\pi_{t:T-1}$

Finite-Horizon Deterministic Optimal Control

- Finite-horizon deterministic optimal control (DOC) problem:

$$\begin{aligned}
\min_{\mathbf{u}_{\tau:T-1}} \quad & V_{\tau}^{\mathbf{u}_{\tau:T-1}}(\mathbf{x}_{\tau}) := \gamma^{T-\tau}\mathfrak{q}(\mathbf{x}_T) + \sum_{t=\tau}^{T-1} \gamma^{t-\tau}\ell_t(\mathbf{x}_t, \mathbf{u}_t) \\
\text{s.t.} \quad & \mathbf{x}_{t+1} = f(\mathbf{x}_t, \mathbf{u}_t), \qquad t = \tau, \ldots, T-1 \\
& \mathbf{x}_t \in \mathcal{X}, \;\; \mathbf{u}_t \in \mathcal{U}
\end{aligned}$$

- An open-loop control sequence $\mathbf{u}^*_{\tau:T-1}$ is optimal for the DOC problem
- The DOC problem is equivalent to the deterministic shortest path (DSP) problem, which led to the forward DP and label correcting algorithms

Infinite-Horizon Stochastic Optimal Control

- In this lecture, we consider what happens with the stochastic optimal control problem as the planning horizon T goes to infinity
- We will consider two formulations of the infinite-horizon stochastic optimal control problem
  - **Discounted Problem**: obtained by letting $T \to \infty$ in the finite-horizon stochastic optimal control problem
  - **First-Exit Problem**: obtained by considering stochastic transitions in the shortest path problem
- Just like the DOC and DSP problems, the discounted problem and the first-exit problem are equivalent, i.e., one can be converted into the other

Discounted Problem

- Let $T \to \infty$ in the finite-horizon stochastic optimal control problem
- The terminal cost $\mathfrak{q}$ is no longer necessary since the problem never terminates
- Assume the motion model p_f and the stage cost ℓ are time-invariant
- The discount factor γ must be < 1 to ensure that the infinite sum of stage costs is finite
- As $T \to \infty$, the time-invariant motion model and stage costs lead to **time-invariant** optimal value function $V^*(\mathbf{x}) = \min_{\pi} V^{\pi}(\mathbf{x})$ and associated optimal policy $\pi^*(\mathbf{x}) = \arg\min_{\pi} V^{\pi}(\mathbf{x})$
- **Discounted Problem**: infinite-horizon stochastic optimal control:

$$
\begin{aligned}
V^*(\mathbf{x}) = \min_{\pi} V^{\pi}(\mathbf{x}) &:= \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t \ell(\mathbf{x}_t, \pi(\mathbf{x}_t)) \;\middle|\; \mathbf{x}_0 = \mathbf{x}\right] \\
\text{s.t.} \;\; & \mathbf{x}_{t+1} \sim p_f(\cdot \mid \mathbf{x}_t, \pi(\mathbf{x}_t)), \\
& \mathbf{x}_t \in \mathcal{X}, \;\; \pi(\mathbf{x}_t) \in \mathcal{U}
\end{aligned}
$$

First-Exit Problem

- Consider a stochastic shortest path problem with state space $\mathcal{X}$ and transitions defined by $p_f(\mathbf{x}'|\mathbf{x},\mathbf{u})$ with control $\mathbf{u} \in \mathcal{U}$
- Let $\mathcal{T} \subseteq \mathcal{X}$ be a set of **terminal states** with terminal cost $\mathfrak{q}(\mathbf{x})$ for $\mathbf{x} \in \mathcal{T}$
- **First-Exit Time**: terminate at $T := \min\{t \geq 0 \mid \mathbf{x}_t \in \mathcal{T}\}$, the first passage time from an initial state $\mathbf{x}_0$ to a terminal state $\mathbf{x}_t \in \mathcal{T}$
- Note that T is a **random variable** unlike in the finite-horizon problem
- **First-Exit Problem**:

$$
\begin{aligned}
V^*(\mathbf{x}) = \min_{\pi} \;\; & V^{\pi}(\mathbf{x}) := \mathbb{E}\left[\mathfrak{q}(\mathbf{x}_T) + \sum_{t=0}^{T-1} \ell(\mathbf{x}_t, \pi(\mathbf{x}_t)) \;\middle|\; \mathbf{x}_0 = \mathbf{x}\right] \\
\text{s.t.} \;\; & \mathbf{x}_{t+1} \sim p_f(\cdot \mid \mathbf{x}_t, \pi(\mathbf{x}_t)), \\
& \mathbf{x}_t \in \mathcal{X}, \;\; \pi(\mathbf{x}_t) \in \mathcal{U}
\end{aligned}
$$

From Discounted Problem to First-Exit Problem

- Given a Discounted Problem, we will define an equivalent First-Exit problem
- **Discounted Problem**: $\mathcal{X}$, $\mathcal{U}$, $p_f(\mathbf{x}'|\mathbf{x},\mathbf{u})$, $\ell(\mathbf{x},\mathbf{u})$
- **First-Exit Problem**:
  - State space: $\tilde{\mathcal{X}} = \mathcal{X} \cup \{\tau\}$ and $\mathcal{T} = \{\tau\}$ where τ is a virtual terminal state
  - Control space: $\tilde{\mathcal{U}} = \mathcal{U}$
  - Motion model:

$$
\begin{aligned}
\tilde{p}_f(\mathbf{x}' \mid \mathbf{x}, \mathbf{u}) &= \gamma p_f(\mathbf{x}' \mid \mathbf{x}, \mathbf{u}) && \text{for } \mathbf{x}' \neq \tau \\
\tilde{p}_f(\tau \mid \mathbf{x}, \mathbf{u}) &= 1 - \gamma, \\
\tilde{p}_f(\mathbf{x}' \mid \tau, \mathbf{u}) &= 0, && \text{for } \mathbf{x}' \neq \tau \\
\tilde{p}_f(\tau \mid \tau, \mathbf{u}) &= 1,
\end{aligned}
$$

  - Stage cost: $\tilde{\ell}(\mathbf{x},\mathbf{u}) = \begin{cases} \ell(\mathbf{x},\mathbf{u}) & \mathbf{x} \neq \tau \\ 0 & \mathbf{x} = \tau \end{cases}$
  - Terminal cost: $\tilde{\mathfrak{q}}(\mathbf{x}) = 0$
  - There is a one-to-one mapping between a policy $\tilde{\pi}$ of this first-exit problem and a policy π of the discounted problem:

$$
\tilde{\pi}(\mathbf{x}) = \begin{cases} \pi(\mathbf{x}) & \mathbf{x} \neq \tau \\ \text{some } \mathbf{u} \in \mathcal{U}, & \mathbf{x} = \tau \end{cases}
$$

From Discounted Problem to First-Exit Problem

- Next, we show that for all $\mathbf{x} \in \mathcal{X}$:

$$\tilde{V}^{\tilde{\pi}}(\mathbf{x}) = \mathbb{E}\left[\sum_{t=0}^{T-1} \tilde{\ell}(\tilde{\mathbf{x}}_t, \tilde{\pi}_t(\tilde{\mathbf{x}}_t)) \;\middle|\; \tilde{\mathbf{x}}_0 = \mathbf{x}\right] = V^{\pi}(\mathbf{x}) = \mathbb{E}\left[\sum_{t=0}^{T-1} \gamma^t \ell(\mathbf{x}_t, \pi_t(\mathbf{x}_t)) \;\middle|\; \mathbf{x}_0 = \mathbf{x}\right]$$

  where the expectations are over $\tilde{\mathbf{x}}_{1:T}$ and $\mathbf{x}_{1:T}$ and subject to transitions induced by $\tilde{\pi}$ and π, respectively

- **Conclusion**: since $\tilde{V}^{\tilde{\pi}}(\mathbf{x}) = V^{\pi}(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}$ and $\tilde{\pi}$ maps to π, by solving the auxiliary first-exit problem, we can obtain an optimal policy and the optimal value for the discounted problem

From Discounted Problem to First-Exit Problem

$$
\begin{aligned}
\mathbb{E}_{\tilde{\mathbf{x}}_{1:T}}[\tilde{\ell}(\tilde{\mathbf{x}}_t,\tilde{\pi}_t(\tilde{\mathbf{x}}_t)) \mid \mathbf{x}_0 = \mathbf{x}] &= \sum_{\bar{\mathbf{x}}_{1:T} \in \tilde{\mathcal{X}}^T} \tilde{\ell}(\bar{\mathbf{x}}_t, \tilde{\pi}_t(\bar{\mathbf{x}}_t))\mathbb{P}(\tilde{\mathbf{x}}_{1:T} = \bar{\mathbf{x}}_{1:T} \mid \mathbf{x}_0 = \mathbf{x}) \\
&= \sum_{\bar{\mathbf{x}}_t \in \tilde{\mathcal{X}}} \tilde{\ell}(\bar{\mathbf{x}}_t, \tilde{\pi}_t(\bar{\mathbf{x}}_t))\mathbb{P}(\tilde{\mathbf{x}}_t = \bar{\mathbf{x}}_t \mid \mathbf{x}_0 = \mathbf{x}) \\
&\overset{\tilde{\ell}(\tau,\mathbf{u})=0}{\underset{\tilde{\mathcal{X}}=\mathcal{X}\cup\{\tau\}}{=\!=\!=\!=}} \sum_{\bar{\mathbf{x}}_t \in \mathcal{X}} \tilde{\ell}(\bar{\mathbf{x}}_t, \tilde{\pi}_t(\bar{\mathbf{x}}_t))\mathbb{P}(\tilde{\mathbf{x}}_t = \bar{\mathbf{x}}_t, \tilde{\mathbf{x}}_t \neq \tau \mid \mathbf{x}_0 = \mathbf{x}) \\
&= \sum_{\bar{\mathbf{x}}_t \in \mathcal{X}} \tilde{\ell}(\bar{\mathbf{x}}_t, \tilde{\pi}_t(\bar{\mathbf{x}}_t))\mathbb{P}(\tilde{\mathbf{x}}_t = \bar{\mathbf{x}}_t \mid \mathbf{x}_0 = \mathbf{x}, \tilde{\mathbf{x}}_t \neq \tau)\mathbb{P}(\tilde{\mathbf{x}}_t \neq \tau \mid \mathbf{x}_0 = \mathbf{x}) \\
&\overset{(?)}{=\!=} \sum_{\bar{\mathbf{x}}_t \in \mathcal{X}} \tilde{\ell}(\bar{\mathbf{x}}_t, \tilde{\pi}_t(\bar{\mathbf{x}}_t))\mathbb{P}(\mathbf{x}_t = \bar{\mathbf{x}}_t \mid \mathbf{x}_0 = \mathbf{x})\gamma^t \\
&= \sum_{\bar{\mathbf{x}}_t \in \mathcal{X}} \ell(\bar{\mathbf{x}}_t, \pi_t(\bar{\mathbf{x}}_t))\mathbb{P}(\mathbf{x}_t = \bar{\mathbf{x}}_t \mid \mathbf{x}_0 = \mathbf{x})\gamma^t \\
&= \mathbb{E}_{\mathbf{x}_{1:T}}\left[\gamma^t \ell(\mathbf{x}_t, \pi_t(\mathbf{x}_t)) \mid \mathbf{x}_0 = \mathbf{x}\right]
\end{aligned}
$$

From Discounted Problem to First-Exit Problem

(?) Show that for transitions $\tilde{p}_f(\mathbf{x}' \mid \mathbf{x}, \mathbf{u})$ under $\tilde{\pi}$, $\mathbb{P}(\tilde{\mathbf{x}}_t \neq 0 \mid \mathbf{x}_0 = \mathbf{x}) = \gamma^t$

- For any $\mathbf{x} \in \mathcal{X}$ and $\mathbf{u} \in \tilde{\mathcal{U}}$:

$$\mathbb{P}(\tilde{\mathbf{x}}_{t+1} \neq \tau \mid \tilde{\mathbf{x}}_t = \mathbf{x}) = 1 - \tilde{p}_f(\tau \mid \mathbf{x}, \mathbf{u}) = \gamma$$

- Similarly, for any $\mathbf{x} \in \mathcal{X}$

$$\begin{aligned}
\mathbb{P}(\tilde{\mathbf{x}}_{t+2} \neq \tau \mid \tilde{\mathbf{x}}_t = \mathbf{x}) &= \sum_{\mathbf{x}' \in \mathcal{X}} \mathbb{P}(\tilde{\mathbf{x}}_{t+2} \neq \tau \mid \tilde{\mathbf{x}}_{t+1} = \mathbf{x}', \tilde{\mathbf{x}}_t = \mathbf{x}) \mathbb{P}(\tilde{\mathbf{x}}_{t+1} = \mathbf{x}' \mid \tilde{\mathbf{x}}_t = \mathbf{x}) \\
&= \sum_{\mathbf{x}' \in \mathcal{X}} \mathbb{P}(\tilde{\mathbf{x}}_{t+2} \neq \tau \mid \tilde{\mathbf{x}}_{t+1} = \mathbf{x}') \mathbb{P}(\tilde{\mathbf{x}}_{t+1} = \mathbf{x}' \mid \tilde{\mathbf{x}}_t = \mathbf{x}) \\
&= \gamma \sum_{\mathbf{x}' \in \mathcal{X}} \tilde{p}_f(\mathbf{x}' \mid \mathbf{x}, \tilde{\pi}(\mathbf{x})) = \gamma^2
\end{aligned}$$

- Similarly, we can show that for any $m > 0$: $\mathbb{P}(\tilde{\mathbf{x}}_{t+m} \neq \tau \mid \mathbf{x}_t = \mathbf{x}) = \gamma^m$

From Discounted Problem to First-Exit Problem

(?) Show that $\mathbb{P}(\tilde{\mathbf{x}}_t = \bar{\mathbf{x}}_t \mid \mathbf{x}_0 = \mathbf{x}, \tilde{\mathbf{x}}_t \neq \tau) = \mathbb{P}(\mathbf{x}_t = \bar{\mathbf{x}}_t \mid \mathbf{x}_0 = \mathbf{x})$

- For any $\mathbf{x}, \mathbf{x}' \in \mathcal{X}$ and $\mathbf{u} = \tilde{\pi}_t(\mathbf{x}) = \pi_t(\mathbf{x})$, we have

$$\mathbb{P}(\tilde{\mathbf{x}}_{t+1} = \mathbf{x}' \mid \tilde{\mathbf{x}}_{t+1} \neq \tau, \tilde{\mathbf{x}}_t = \mathbf{x}, \tilde{\mathbf{u}}_t = \mathbf{u}) = \frac{\mathbb{P}(\tilde{\mathbf{x}}_{t+1} = \mathbf{x}', \tilde{\mathbf{x}}_{t+1} \neq \tau \mid \tilde{\mathbf{x}}_t = \mathbf{x}, \tilde{\mathbf{u}}_t = \mathbf{u})}{\mathbb{P}(\tilde{\mathbf{x}}_{t+1} \neq \tau \mid \tilde{\mathbf{x}}_t = \mathbf{x}, \tilde{\mathbf{u}}_t = \mathbf{u})}$$
$$= \frac{\tilde{p}_f(\mathbf{x}' \mid \mathbf{x}, \mathbf{u})}{\gamma} = p_f(\mathbf{x}' \mid \mathbf{x}, \mathbf{u}) = \mathbb{P}(\mathbf{x}_{t+1} = \mathbf{x}' \mid \mathbf{x}_t = \mathbf{x}, \mathbf{u}_t = \mathbf{u})$$

- Similarly, it can be shown that for $\bar{\mathbf{x}}_t \in \mathcal{X}$:

$$\mathbb{P}(\tilde{\mathbf{x}}_t = \bar{\mathbf{x}}_t \mid \mathbf{x}_0 = \mathbf{x}, \tilde{\mathbf{x}}_t \neq 0) = \mathbb{P}(\mathbf{x}_t = \bar{\mathbf{x}}_t \mid \mathbf{x}_0 = \mathbf{x})$$

Outline

Bellman Equation

- Recall the Dynamic Programming algorithm for finite horizon T:

$$V_T(\mathbf{x}) = \mathfrak{q}(\mathbf{x}), \quad \forall \mathbf{x} \in \mathcal{X}$$
$$V_t(\mathbf{x}) = \min_{\mathbf{u} \in \mathcal{U}(\mathbf{x})} \ell(\mathbf{x}, \mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\mathbf{u})} \left[V_{t+1}(\mathbf{x}') \right], \quad \forall \mathbf{x} \in \mathcal{X}, t = T-1, \ldots, \tau$$

- **Bellman Equation**: as $T \to \infty$, the sequence $\ldots, V_{t+1}(\mathbf{x}), V_t(\mathbf{x}), \ldots$ converges to a fixed point $V(\mathbf{x})$ of the dynamic programming recursion:

$$V(\mathbf{x}) = \min_{\mathbf{u} \in \mathcal{U}(\mathbf{x})} \left\{ \ell(\mathbf{x}, \mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\mathbf{u})} \left[V(\mathbf{x}') \right] \right\}, \quad \forall \mathbf{x} \in \mathcal{X}$$

- Assuming convergence, $V(\mathbf{x})$ is equal to the optimal value $V^*(\mathbf{x})$
- Both $V^*(\mathbf{x})$ and the associated opitmal policy $\pi^*(\mathbf{x})$ are **stationary**
- The Bellman Equation needs to be solved for all $\mathbf{x} \in \mathcal{X}$ simultaneously, which can be done analytically only for very few problems (e.g., the Linear Quadratic Regulator (LQR) problem)

Bellman Equation

- The optimal value function $V^*(\mathbf{x})$ satisfies:

$$V^*(\mathbf{x}) = \min_{\mathbf{u}\in\mathcal{U}(\mathbf{x})} \left\{\ell(\mathbf{x},\mathbf{u}) + \gamma\mathbb{E}_{\mathbf{x}'\sim p_f(\cdot|\mathbf{x},\pi(\mathbf{x}))}\left[V^*(\mathbf{x}')\right]\right\}, \quad \forall \mathbf{x}\in\mathcal{X}$$

- The value function $V^\pi(\mathbf{x})$ of policy π satisfies:

$$V^\pi(\mathbf{x}) = \ell(\mathbf{x},\pi(\mathbf{x})) + \gamma\mathbb{E}_{\mathbf{x}'\sim p_f(\cdot|\mathbf{x},\pi(\mathbf{x}))}\left[V^\pi(\mathbf{x}')\right], \quad \forall \mathbf{x}\in\mathcal{X}$$

- The latter can be obtained from:

$$\begin{aligned} V^\pi(\mathbf{x}) :=& \mathbb{E}\left[\sum_{t=0}^{\infty}\gamma^t\ell(\mathbf{x}_t,\pi(\mathbf{x}_t)) \,\middle|\, \mathbf{x}_0=\mathbf{x}\right] \\ =& \ell(\mathbf{x},\pi(\mathbf{x})) + \gamma\mathbb{E}\left[\sum_{t=1}^{\infty}\gamma^{t-1}\ell(\mathbf{x}_t,\pi(\mathbf{x}_t)) \,\middle|\, \mathbf{x}_0=\mathbf{x}\right] \\ =& \ell(\mathbf{x},\pi(\mathbf{x})) + \gamma\mathbb{E}_{\mathbf{x}'\sim p_f(\cdot|\mathbf{x},\pi(\mathbf{x}))}\left[V^\pi(\mathbf{x}')\right] \end{aligned}$$

Action-Value (Q) Function

- **Value Function** $V^\pi(\mathbf{x})$: the expected long-term cost of following policy π starting from state $\mathbf{x}$
- **Q Function** $Q^\pi(\mathbf{x},\mathbf{u})$: the expected long-term cost of taking action $\mathbf{u}$ in state $\mathbf{x}$ and following policy π afterwards:

$$\begin{aligned} Q^\pi(\mathbf{x},\mathbf{u}) &:= \ell(\mathbf{x},\mathbf{u}) + \mathbb{E}\left[\sum_{t=1}^{\infty} \gamma^t \ell(\mathbf{x}_t, \pi(\mathbf{x}_t)) \,\middle|\, \mathbf{x}_0 = \mathbf{x}\right] \\ &= \ell(\mathbf{x},\mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\mathbf{u})}\left[V^\pi(\mathbf{x}')\right] \\ &= \ell(\mathbf{x},\mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\mathbf{u})}\underbrace{\left[Q^\pi(\mathbf{x}', \pi(\mathbf{x}'))\right]}_{V^\pi(\mathbf{x}')} \end{aligned}$$

- **Optimal Q Function**: $Q^*(\mathbf{x},\mathbf{u}) := \min_\pi Q^\pi(\mathbf{x},\mathbf{u})$

$$\begin{aligned} Q^*(\mathbf{x},\mathbf{u}) &= \ell(\mathbf{x},\mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\mathbf{u})}\left[V^*(\mathbf{x}')\right] \\ &= \ell(\mathbf{x},\mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\mathbf{u})}\left[\min_{\mathbf{u}' \in \mathcal{U}(\mathbf{x}')} Q^*(\mathbf{x}',\mathbf{u}')\right] \\ \pi^*(\mathbf{x}) &\in \arg\min_{\mathbf{u} \in \mathcal{U}} Q^*(\mathbf{x},\mathbf{u}) \end{aligned}$$

Bellman Operators

- **Hamiltonian**:

$$H[\mathbf{x},\mathbf{u},V] = \ell(\mathbf{x},\mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\mathbf{u})}\left[V(\mathbf{x}')\right]$$

- **Policy Evaluation Operator**:

$$\mathcal{B}_\pi[V](\mathbf{x}) := \ell(\mathbf{x},\pi(\mathbf{x})) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\pi(\mathbf{x}))}\left[V(\mathbf{x}')\right] \; = H[\mathbf{x},\pi(\mathbf{x}),V(\cdot)]$$

- **Value Operator**:

$$\mathcal{B}_*[V](\mathbf{x}) := \min_{\mathbf{u}\in\mathcal{U}} \left\{\ell(\mathbf{x},\mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\mathbf{u})}\left[V(\mathbf{x}')\right]\right\} \; = \min_{\mathbf{u}\in\mathcal{U}} H[\mathbf{x},\mathbf{u},V(\cdot)]$$

- **Policy Q-Evaluation Operator**:

$$\mathcal{B}_\pi[Q](\mathbf{x},\mathbf{u}) := \ell(\mathbf{x},\mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\mathbf{u})}\left[Q(\mathbf{x}',\pi(\mathbf{x}'))\right] \; = H[\mathbf{x},\mathbf{u},Q(\cdot,\pi(\cdot))]$$

- **Q-Value Operator**:

$$\mathcal{B}_*[Q](\mathbf{x},\mathbf{u}) := \ell(\mathbf{x},\mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\mathbf{u})}\left[\min_{\mathbf{u}'\in\mathcal{U}} Q(\mathbf{x}',\mathbf{u}')\right] \; = H[\mathbf{x},\mathbf{u},\min_{\mathbf{u}'\in\mathcal{U}} Q(\cdot,\mathbf{u}')]$$

Finite-Horizon Problem

- Trajectories terminate at fixed $T < \infty$

$$\min_{\pi} V_\tau^\pi(\mathbf{x}) = \mathbb{E}\left[\gamma^{T-\tau}\mathfrak{q}(\mathbf{x}_T) + \sum_{t=\tau}^{T-1}\gamma^{t-\tau}\ell(\mathbf{x}_t, \pi_t(\mathbf{x}_t)) \,\middle|\, \mathbf{x}_\tau = \mathbf{x}\right]$$

- The optimal value $V_t^*(\mathbf{x})$ can be found with a single backward pass through time, initialized from $V_T^*(\mathbf{x}) = \mathfrak{q}(\mathbf{x})$ and following the recursion:

Bellman Equations (Finite-Horizon Problem)

Hamiltonian: $H[\mathbf{x}, \mathbf{u}, V(\cdot)] = \ell(\mathbf{x}, \mathbf{u}) + \gamma\mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\mathbf{u})}[V(\mathbf{x}')]$

Policy Evaluation: $V_t^\pi(\mathbf{x}) = Q_t^\pi(\mathbf{x}, \pi_t(\mathbf{x})) = H[\mathbf{x}, \pi_t(\mathbf{x}), V_{t+1}^\pi(\cdot)]$

Bellman Equation: $V_t^*(\mathbf{x}) = \min_{\mathbf{u}\in\mathcal{U}} Q_t^*(\mathbf{x}, \mathbf{u}) = \min_{\mathbf{u}\in\mathcal{U}} H[\mathbf{x}, \mathbf{u}, V_{t+1}^*(\cdot)]$

Optimal Policy: $\pi_t^*(\mathbf{x}) = \arg\min_{\mathbf{u}\in\mathcal{U}} Q_t^*(\mathbf{x}, \mathbf{u}) = \arg\min_{\mathbf{u}\in\mathcal{U}} H[\mathbf{x}, \mathbf{u}, V_{t+1}^*(\cdot)]$

Discounted Problem

- Trajectories continue forever but costs are discounted via $\gamma \in [0, 1)$:

$$\min_{\pi} V^{\pi}(\mathbf{x}) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t \ell(\mathbf{x}_t, \pi(\mathbf{x}_t)) \middle| \mathbf{x}_0 = \mathbf{x}\right]$$

Bellman Equations (Discounted Problem)

Hamiltonian: $H[\mathbf{x}, \mathbf{u}, V(\cdot)] = \ell(\mathbf{x}, \mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\mathbf{u})}[V(\mathbf{x}')]$

Policy Evaluation: $V^{\pi}(\mathbf{x}) = Q^{\pi}(\mathbf{x}, \pi(\mathbf{x})) = H[\mathbf{x}, \pi(\mathbf{x}), V^{\pi}(\cdot)]$

Bellman Equation: $V^*(\mathbf{x}) = \min_{\mathbf{u} \in \mathcal{U}} Q^*(\mathbf{x}, \mathbf{u}) = \min_{\mathbf{u} \in \mathcal{U}} H[\mathbf{x}, \mathbf{u}, V^*(\cdot)]$

Optimal Policy: $\pi^*(\mathbf{x}) = \arg\min_{\mathbf{u} \in \mathcal{U}} Q^*(\mathbf{x}, \mathbf{u}) = \arg\min_{\mathbf{u} \in \mathcal{U}} H[\mathbf{x}, \mathbf{u}, V^*(\cdot)]$

First-Exit Problem

- Trajectories terminate at $T := \inf\{t \geq 1 | \mathbf{x}_t \in \mathcal{T}\}$, the first passage time from initial state $\mathbf{x}_0$ to a terminal state $\mathbf{x}_t \in \mathcal{T} \subseteq \mathcal{X}$:

$$\min_{\pi} V^{\pi}(\mathbf{x}) = \mathbb{E}\left[\mathfrak{q}(\mathbf{x}_T) + \sum_{t=0}^{T-1} \ell(\mathbf{x}_t, \pi(\mathbf{x}_t)) \middle| \mathbf{x}_0 = \mathbf{x}\right]$$

- At terminal states, $V^*(\mathbf{x}) = V^{\pi}(\mathbf{x}) = \mathfrak{q}(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{T}$
- At other states, the following are satisfied:

Bellman Equations (First-Exit Problem)

Hamiltonian: $H[\mathbf{x}, \mathbf{u}, V(\cdot)] = \ell(\mathbf{x}, \mathbf{u}) + \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\mathbf{u})}[V(\mathbf{x}')]$

Policy Evaluation: $V^{\pi}(\mathbf{x}) = Q^{\pi}(\mathbf{x}, \pi(\mathbf{x})) = H[\mathbf{x}, \pi(\mathbf{x}), V^{\pi}(\cdot)]$

Bellman Equation: $V^*(\mathbf{x}) = \min_{\mathbf{u} \in \mathcal{U}} Q^*(\mathbf{x}, \mathbf{u}) = \min_{\mathbf{u} \in \mathcal{U}} H[\mathbf{x}, \mathbf{u}, V^*(\cdot)]$

Optimal Policy: $\pi^*(\mathbf{x}) = \arg\min_{\mathbf{u} \in \mathcal{U}} Q^*(\mathbf{x}, \mathbf{u}) = \arg\min_{\mathbf{u} \in \mathcal{U}} H[\mathbf{x}, \mathbf{u}, V^*(\cdot)]$

Bellman Equation Algorithms

- To the determine the value function $V^{\pi}(\mathbf{x})$ of policy π in either the Discounted or First-Exit Problem, we need to solve a policy evaluation equation:
  - Policy Evaluation: $V^{\pi}(\mathbf{x}) = H[\mathbf{x}, \pi(\mathbf{x}), V^{\pi}(\cdot)]$
  - Policy Q-Evaluation: $Q^{\pi}(\mathbf{x}, \mathbf{u}) = H[\mathbf{x}, \mathbf{u}, Q^{\pi}(\cdot, \pi(\cdot))]$
- A policy evaluation equation can be solved by:
  - Iterative Policy Evaluation
  - Linear System Solution (only for finite state space $\mathcal{X}$)
- To the determine the optimal value function $V^{*}(\mathbf{x})$ in either the Discounted or First-Exit Problem, we need to solve a Bellman equation:
  - Bellman Equation: $V^{*}(\mathbf{x}) = \min_{\mathbf{u} \in \mathcal{U}} H[\mathbf{x}, \mathbf{u}, V^{*}(\cdot)]$
  - Q-Bellman Equation: $Q^{*}(\mathbf{x}, \mathbf{u}) = H[\mathbf{x}, \mathbf{u}, \min_{\mathbf{u}' \in \mathcal{U}} Q^{*}(\cdot, \mathbf{u}')]$
- A Bellman equation can be solved by:
  - Value Iteration
  - Policy Iteration
  - Linear Programming (only for finite state space $\mathcal{X}$)

Outline

Policy Evaluation

Policy Evaluation Theorem (Discounted Problem)

The value function $V^\pi(\mathbf{x})$ of policy π is the unique solution of:

$$V^\pi(\mathbf{x}) = \ell(\mathbf{x}, \pi(\mathbf{x})) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\pi(\mathbf{x}))}\left[V^\pi(\mathbf{x}')\right], \qquad \forall \mathbf{x} \in \mathcal{X}.$$

If $\gamma \in [0,1)$, for any initial condition $V_0(\mathbf{x})$, the sequence $V_k(\mathbf{x})$ generated by the recursion below converges to $V^\pi(\mathbf{x})$:

$$V_{k+1}(\mathbf{x}) = \ell(\mathbf{x}, \pi(\mathbf{x})) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\pi(\mathbf{x}))}\left[V_k(\mathbf{x}')\right], \qquad \forall \mathbf{x} \in \mathcal{X}.$$

- The PE algorithm requires infinite iterations for $V_k(\mathbf{x})$ to converge to $V^\pi(\mathbf{x})$
- In practice, the PE algorithm is terminated when $|V_{k+1}(\mathbf{x}) - V_k(\mathbf{x})| < \epsilon$ for all $\mathbf{x} \in \mathcal{X}$ and some threshold ϵ

Policy Evaluation

- **Proper policy for first-exit problem**: a policy π for which there exists an integer m such that $\mathbb{P}(\mathbf{x}_m \in \mathcal{T} \mid \mathbf{x}_0 = \mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{X} \setminus \mathcal{T}$

Policy Evaluation Theorem (First-Exit Problem)

The value function $V^\pi(\mathbf{x})$ of policy π is the unique solution of:

$$
\begin{aligned}
V^\pi(\mathbf{x}) &= \mathfrak{q}(\mathbf{x}), && \forall \mathbf{x} \in \mathcal{T}, \\
V^\pi(\mathbf{x}) &= \ell(\mathbf{x}, \pi(\mathbf{x})) + \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\pi(\mathbf{x}))}\left[V^\pi(\mathbf{x}')\right], && \forall \mathbf{x} \in \mathcal{X} \setminus \mathcal{T}.
\end{aligned}
$$

If π is a proper policy, for any initial condition $V_0(\mathbf{x})$, the sequence $V_k(\mathbf{x})$ generated by the recursion below converges to $V^\pi(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X} \setminus \mathcal{T}$:

$$
V_{k+1}(\mathbf{x}) = \ell(\mathbf{x}, \pi(\mathbf{x})) + \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\pi(\mathbf{x}))}\left[V_k(\mathbf{x}')\right], \qquad \forall \mathbf{x} \in \mathcal{X} \setminus \mathcal{T}.
$$

Policy Evaluation (Discounted Finite-State Problem)

- Let $\mathcal{X} = \{1, \ldots, n\}$
- Let $\mathbf{v}_i := V^{\pi}(i)$, $\boldsymbol{\ell}_i := \ell(i, \pi(i))$, $P_{ij} := p_f(j \mid i, \pi(i))$ for $i, j = 1, \ldots, n$
- Policy evaluation:

$$\mathbf{v} = \boldsymbol{\ell} + \gamma P \mathbf{v} \qquad \Rightarrow \qquad (I - \gamma P)\mathbf{v} = \boldsymbol{\ell}$$

- Existence of solution: The matrix P has eigenvalues with modulus ≤ 1. All eigenvalues of γP have modulus < 1, so $(\gamma P)^T \to 0$ as $T \to \infty$ and $(I - \gamma P)^{-1}$ exists.
- The Policy Evaluation Theorem is an iterative solution to the linear system:

$$\begin{aligned}
\mathbf{v}_1 &= \boldsymbol{\ell} + \gamma P \mathbf{v}_0 \\
\mathbf{v}_2 &= \boldsymbol{\ell} + \gamma P \mathbf{v}_1 = \boldsymbol{\ell} + \gamma P \boldsymbol{\ell} + (\gamma P)^2 \mathbf{v}_0 \\
&\vdots \\
\mathbf{v}_k &= (I + \gamma P + (\gamma P)^2 + \ldots + (\gamma P)^{k-1})\boldsymbol{\ell} + (\gamma P)^k \mathbf{v}_0 \\
&\vdots \\
\mathbf{v}_\infty &\to (I - \gamma P)^{-1} \boldsymbol{\ell}
\end{aligned}$$

Policy Evaluation (First-Exit Finite-State Problem)

- Let $\mathcal{X} = \mathcal{N} \cup \mathcal{T}$ and $P_{ij} := p_f(j \mid i, \pi(i))$ for $i, j \in \mathcal{N} \cup \mathcal{T}$
- Let $\mathbf{q}_i := \mathfrak{q}(i)$ for $i \in \mathcal{T}$ and $\mathbf{v}_i := V^\pi(i)$, $\boldsymbol{\ell}_i := \ell(i, \pi(i))$ for $i \in \mathcal{N}$
- Policy evaluation:

$$\mathbf{v} = \boldsymbol{\ell} + P_{\mathcal{N}\mathcal{N}}\mathbf{v} + P_{\mathcal{N}\mathcal{T}}\mathbf{q} \qquad \Rightarrow \qquad (I - P_{\mathcal{N}\mathcal{N}})\,\mathbf{v} = \boldsymbol{\ell} + P_{\mathcal{N}\mathcal{T}}\mathbf{q}$$

- Existence of solution: A unique solution for $\mathbf{v}$ exists as long as π is a proper policy. By the Chapman-Kolmogorov equation, $[P^k]_{ij} = \mathbb{P}(\mathbf{x}_k = j \mid \mathbf{x}_0 = i)$ and since π is proper, $[P^k]_{ij} \to 0$ as $k \to \infty$ for all $i, j \in \mathcal{X} \setminus \mathcal{T}$. Since $P^k_{\mathcal{N}\mathcal{N}}$ vanishes as $k \to \infty$, all eigenvalues of $P_{\mathcal{N}\mathcal{N}}$ must have modulus less than 1 and $(I - P_{\mathcal{N}\mathcal{N}})^{-1}$ exists.
- The Policy Evaluation Theorem is an iterative solution to the linear system:

$$\begin{aligned}
\mathbf{v}_1 &= \boldsymbol{\ell} + P_{\mathcal{N}\mathcal{T}}\mathbf{q} + P_{\mathcal{N}\mathcal{N}}\mathbf{v}_0 \\
\mathbf{v}_2 &= \boldsymbol{\ell} + P_{\mathcal{N}\mathcal{T}}\mathbf{q} + P_{\mathcal{N}\mathcal{N}}\mathbf{v}_1 = \boldsymbol{\ell} + P_{\mathcal{N}\mathcal{T}}\mathbf{q} + P_{\mathcal{N}\mathcal{N}}\left(\boldsymbol{\ell} + P_{\mathcal{N}\mathcal{T}}\mathbf{q}\right) + P^2_{\mathcal{N}\mathcal{N}}\mathbf{v}_0 \\
\mathbf{v}_\infty &\to (I - P_{\mathcal{N}\mathcal{N}})^{-1}\left(\boldsymbol{\ell} + P_{\mathcal{N}\mathcal{T}}\mathbf{q}\right)
\end{aligned}$$

Outline

Value Iteration

Value Iteration Theorem (Discounted Problem)

The optimal value function $V^*(\mathbf{x})$ is the unique solution of:

$$V^*(\mathbf{x}) = \min_{\mathbf{u} \in \mathcal{U}} \left\{ \ell(\mathbf{x}, \mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\mathbf{u})} \left[V^*(\mathbf{x}') \right] \right\}, \qquad \forall \mathbf{x} \in \mathcal{X}.$$

If $\gamma \in [0, 1)$, for any initial condition $V_0(\mathbf{x})$, the sequence $V_k(\mathbf{x})$ generated by the recursion below converges to $V^*(\mathbf{x})$:

$$V_{k+1}(\mathbf{x}) = \min_{\mathbf{u} \in \mathcal{U}} \left\{ \ell(\mathbf{x}, \mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\mathbf{u})} \left[V_k(\mathbf{x}') \right] \right\}, \qquad \forall \mathbf{x} \in \mathcal{X}.$$

- The VI algorithm is an infinite-horizon equivalent of the DP algorithm ($V_0(\mathbf{x})$ in VI corresponds to $V_{T \to \infty}(\mathbf{x})$ in DP)
- VI requires infinite iterations for $V_k(\mathbf{x})$ to converge to $V^*(\mathbf{x})$
- In practice, the VI algorithm is terminated when $|V_{k+1}(\mathbf{x}) - V_k(\mathbf{x})| < \epsilon$ for all $\mathbf{x} \in \mathcal{X}$ and some threshold ϵ

Gauss-Seidel Value Iteration

- A regular VI implementation stores the values from a previous iteration and updates them for all states simultaneously:

$$\hat{V}(\mathbf{x}) \leftarrow \min_{\mathbf{u}\in\mathcal{U}} \left\{\ell(\mathbf{x},\mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}'\sim p_f(\cdot|\mathbf{x},\mathbf{u})}\left[V(\mathbf{x}')\right]\right\}, \qquad \forall \mathbf{x} \in \mathcal{X}$$

$$V(\mathbf{x}) \leftarrow \hat{V}(\mathbf{x}), \qquad \forall \mathbf{x} \in \mathcal{X}$$

- **Gauss-Seidel Value Iteration** updates the values in place:

$$V(\mathbf{x}) \leftarrow \min_{\mathbf{u}\in\mathcal{U}(\mathbf{x})} \left\{\ell(\mathbf{x},\mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}'\sim p_f(\cdot|\mathbf{x},\mathbf{u})}\left[V(\mathbf{x}')\right]\right\}, \qquad \forall \mathbf{x} \in \mathcal{X}$$

- Gauss-Seidel VI converges and often leads to faster convergence and requires less memory than VI

Value Iteration

Value Iteration Theorem (First-Exit Problem)

The optimal value function $V^*(\mathbf{x})$ is the unique solution of:

$$\begin{aligned} V^*(\mathbf{x}) &= \mathfrak{q}(\mathbf{x}), && \forall \mathbf{x} \in \mathcal{T}, \\ V^*(\mathbf{x}) &= \min_{\mathbf{u} \in \mathcal{U}} \left\{ \ell(\mathbf{x}, \mathbf{u}) + \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\mathbf{u})} \left[V^*(\mathbf{x}') \right] \right\}, && \forall \mathbf{x} \in \mathcal{X} \setminus \mathcal{T}. \end{aligned}$$

If a proper policy exists, for any initial condition $V_0(\mathbf{x})$, the sequence $V_k(\mathbf{x})$ generated by the recursion below converges to $V^*(\mathbf{x})$:

$$\begin{aligned} V_k(\mathbf{x}) &= \mathfrak{q}(\mathbf{x}), && \forall \mathbf{x} \in \mathcal{T}, \ \forall k, \\ V_{k+1}(\mathbf{x}) &= \min_{\mathbf{u} \in \mathcal{U}} \left\{ \ell(\mathbf{x}, \mathbf{u}) + \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\mathbf{u})} \left[V_k(\mathbf{x}') \right] \right\}, && \forall \mathbf{x} \in \mathcal{X} \setminus \mathcal{T}. \end{aligned}$$

Contraction in Discounted Problems

Contraction Mapping

Let $\mathcal{F}(\mathcal{X})$ denote the linear space of bounded functions $V : \mathcal{X} \mapsto \mathbb{R}$ with norm $\|V\|_\infty := \sup_{\mathbf{x} \in \mathcal{X}} |V(\mathbf{x})|$. A function $\mathcal{B} : \mathcal{F}(\mathcal{X}) \mapsto \mathcal{F}(\mathcal{X})$ is called a *contraction mapping* if there exists a scalar $\alpha < 1$ such that:

$$\|\mathcal{B}[V] - \mathcal{B}[V']\|_\infty \leq \alpha \|V - V'\|_\infty \qquad \forall V, V' \in \mathcal{F}(\mathcal{X})$$

Contraction Mapping Theorem

If $\mathcal{B} : \mathcal{F}(\mathcal{X}) \mapsto \mathcal{F}(\mathcal{X})$ is a contraction mapping, then there exists a unique function $V^* \in \mathcal{F}(\mathcal{X})$ such that $\mathcal{B}[V^*] = V^*$.

Contraction in Discounted Problems

Properties of $\mathcal{B}_*[V]$

The operator $\mathcal{B}_*[V](\mathbf{x}) = \min_{\mathbf{u}\in\mathcal{U}} \left\{ \ell(\mathbf{x},\mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}'\sim p_f(\cdot|\mathbf{x},\mathbf{u})} \left[V(\mathbf{x}') \right] \right\}$ satisfies:

1. Monotonicity: $V(\mathbf{x}) \le V'(\mathbf{x}) \quad \Rightarrow \quad \mathcal{B}_*[V](\mathbf{x}) \le \mathcal{B}_*[V'](\mathbf{x})$
2. γ-Additivity: $\mathcal{B}_*[V + d](\mathbf{x}) = \mathcal{B}_*[V](\mathbf{x}) + \gamma d$ for $d \in \mathbb{R}$
3. Contraction: $\|\mathcal{B}_*[V] - \mathcal{B}_*[V']\|_\infty \le \gamma \|V - V'\|_\infty$

- **Proof of Contraction**: Let $d = \sup_{\mathbf{x}} |V(\mathbf{x}) - V'(\mathbf{x})|$. Then:

$$V(\mathbf{x}) - d \le V'(\mathbf{x}) \le V(\mathbf{x}) + d, \quad \forall \mathbf{x} \in \mathcal{X}$$

Apply $\mathcal{B}_*$ to both sides and use monotonicity and γ-additivity:

$$\mathcal{B}_*[V](\mathbf{x}) - \gamma d \le \mathcal{B}_*[V'](\mathbf{x}) \le \mathcal{B}_*[V](\mathbf{x}) + \gamma d, \quad \forall \mathbf{x} \in \mathcal{X}$$

Proof of VI Convergence in Discounted Problems

- $\mathcal{B}_*[V]$ is monotone, γ-additive, and a contraction mapping
- By the contraction mapping theorem, there exists $V^*(\mathbf{x})$ such that $\mathcal{B}_*[V^*] = V^*$
- Value Iteration Algorithm:

$$V_0(\mathbf{x}) \equiv 0$$
$$V_{k+1}(\mathbf{x}) = \mathcal{B}_*[V_k](\mathbf{x})$$

- Since $\mathcal{B}_*[V]$ is a contraction, the sequence V_k is Cauchy, i.e., $\|V_{k+1} - V_k\|_\infty \leq \gamma^k \|V_1 - V_0\|_\infty$
- If $(\mathcal{F}(\mathcal{X}), \|\cdot\|_\infty)$ is a complete metric space, then V_k has a limit $V^* \in \mathcal{F}(\mathcal{X})$ and V^* is a fixed point of $\mathcal{B}_*$

Outline

Infinite-Horizon Optimal Control

Bellman Equations

Policy Evaluation

Value Iteration

Policy Iteration

Linear Programming

Discounted Problem Policy Iteration (PI)

- PI is an alternative algorithm to VI for computing $V^*(\mathbf{x})$
- PI iterates over policies instead of values
- **Policy Iteration**: repeat until $V^{\pi'}(\mathbf{x}) = V^{\pi}(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}$:
  1. **Policy Evaluation**: given a policy π, compute V^{π}:

$$V^{\pi}(\mathbf{x}) = \ell(\mathbf{x}, \pi(\mathbf{x})) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\mathbf{u})} \left[V^{\pi}(\mathbf{x}') \right], \qquad \forall \mathbf{x} \in \mathcal{X}$$

  2. **Policy Improvement**: given V^{π}, obtain a new policy π':

$$\pi'(\mathbf{x}) \in \arg\min_{\mathbf{u} \in \mathcal{U}} \left\{ \ell(\mathbf{x}, \mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\mathbf{u})} \left[V^{\pi}(\mathbf{x}') \right] \right\}, \qquad \forall \mathbf{x} \in \mathcal{X}$$

First-Exit Problem Policy Iteration (PI)

- **Policy Iteration**: repeat until $V^{\pi'}(\mathbf{x}) = V^{\pi}(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X} \setminus \mathcal{T}$:
  1. **Policy Evaluation**: given a policy π, compute V^{π}:

$$V^{\pi}(\mathbf{x}) = \ell(\mathbf{x}, \pi(\mathbf{x})) + \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\mathbf{u})} \left[V^{\pi}(\mathbf{x}') \right], \qquad \forall \mathbf{x} \in \mathcal{X} \setminus \mathcal{T}$$

  2. **Policy Improvement**: given V^{π}, obtain a new policy π':

$$\pi'(\mathbf{x}) = \arg\min_{\mathbf{u} \in \mathcal{U}} \left\{ \ell(\mathbf{x}, \mathbf{u}) + \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\mathbf{u})} \left[V^{\pi}(\mathbf{x}') \right] \right\}, \qquad \forall \mathbf{x} \in \mathcal{X} \setminus \mathcal{T}$$

Policy Improvement Theorem

Let π and π' be such that $V^\pi(\mathbf{x}) \geq Q^\pi(\mathbf{x}, \pi'(\mathbf{x}))$ for all $\mathbf{x} \in \mathcal{X}$. Then, π' is at least as good as π, i.e., $V^\pi(\mathbf{x}) \geq V^{\pi'}(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}$

- **Proof**:

$$
\begin{aligned}
V^\pi(\mathbf{x}) \geq Q^\pi(\mathbf{x}, \pi'(\mathbf{x})) &= \ell(\mathbf{x}, \pi'(\mathbf{x})) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x}, \pi'(\mathbf{x}))} \left[V^\pi(\mathbf{x}') \right] \\
&\geq \ell(\mathbf{x}, \pi'(\mathbf{x})) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x}, \pi'(\mathbf{x}))} \left[Q^\pi(\mathbf{x}', \pi'(\mathbf{x}')) \right] \\
&= \ell(\mathbf{x}, \pi'(\mathbf{x})) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x}, \pi'(\mathbf{x}))} \left\{ \ell(\mathbf{x}', \pi'(\mathbf{x}')) + \gamma \mathbb{E}_{\mathbf{x}'' \sim p_f(\cdot|\mathbf{x}', \pi'(\mathbf{x}'))} V^\pi(\mathbf{x}'') \right\} \\
&\geq \cdots \geq \mathbb{E} \left[\sum_{t=0}^{\infty} \gamma^t \ell(\mathbf{x}_t, \pi'(\mathbf{x}_t)) \middle| \mathbf{x}_0 = \mathbf{x} \right] = V^{\pi'}(\mathbf{x})
\end{aligned}
$$

Theorem: Optimality of PI

Suppose that $\mathcal{X}$ is finite and:

- $\gamma \in [0, 1)$ (Discounted Problem),
- there exists a proper policy (First-Exit Problem).

Then, the Policy Iteration algorithm converges to an optimal policy after a finite number of steps.

Proof of Optimality of PI (First-Exit Problem)

- Let π be a proper policy with value V^π obtained from Policy Evaluation
- Let π' be the policy obtained from Policy Improvement
- By definition of Policy Improvement: $V^\pi(\mathbf{x}) \geq Q^\pi(\mathbf{x}, \pi'(\mathbf{x}))$ for all $\mathbf{x} \in \mathcal{X} \setminus \mathcal{T}$
- By the Policy Improvement Thm., $V^\pi(\mathbf{x}) \geq V^{\pi'}(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X} \setminus \mathcal{T}$
- Since π is proper, $V^\pi(\mathbf{x}) < \infty$ for all $\mathbf{x} \in \mathcal{X}$, and hence π' is proper
- Since π' is proper, the Policy Evaluation step has a unique solution $V^{\pi'}$
- Since the number of stationary policies is finite, eventually $V^\pi = V^{\pi'}$ after a finite number of steps
- Once V^π has converged, it follows from the Policy Improvement step:

$$V^{\pi'}(\mathbf{x}) = V^\pi(\mathbf{x}) = \min_{\mathbf{u} \in \mathcal{U}} \left\{ \ell(\mathbf{x}, \mathbf{u}) + \sum_{\mathbf{x}' \in \mathcal{X}} \tilde{p}_f(\mathbf{x}' \mid \mathbf{x}, \mathbf{u}) V^\pi(\mathbf{x}') \right\}, \quad \mathbf{x} \in \mathcal{X} \setminus \mathcal{T}$$

- Since this is the Bellman equation for the first-exit problem, we have converged to an optimal policy $\pi^* = \pi$ with optimal value $V^* = V^\pi$

Generalized Policy Iteration

- PI and VI have a lot in common
- Rewrite VI as follows:
  2. **Policy Improvement**: Given $V_k(\mathbf{x})$ obtain a policy:

$$\pi(\mathbf{x}) \in \arg\min_{\mathbf{u}\in\mathcal{U}} \left\{ \ell(\mathbf{x},\mathbf{u}) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\mathbf{u})} \left[V_k(\mathbf{x}') \right] \right\}, \qquad \forall \mathbf{x} \in \mathcal{X}$$

  1. **Value Update**: Given $\pi(\mathbf{x})$ and $V_k(\mathbf{x})$, compute

$$V_{k+1}(\mathbf{x}) = \ell(\mathbf{x},\pi(\mathbf{x})) + \gamma \mathbb{E}_{\mathbf{x}' \sim p_f(\cdot|\mathbf{x},\mathbf{u})} \left[V_k(\mathbf{x}') \right], \qquad \forall \mathbf{x} \in \mathcal{X}$$

- Value Update is a single step of the iterative Policy Evaluation algorithm
- PI solves the Policy Evaluation equation completely, which is equivalent to running the Value Update step of VI an infinite number of times
- **Generalized Policy Iteration**: assuming the Value Update and Policy Improvement steps are executed an infinite number of times for all states, all combinations of the following converge:
  - Any number of Value Update steps in between Policy Improvement steps
  - Any number of states updated at each Value Update step
  - Any number of states updated at each Policy Improvement step

Complexity of VI and PI

- Consider the complexity of VI and PI for a finite state space $\mathcal{X}$
- **Complexity of VI per Iteration**: $O(|\mathcal{X}|^2|\mathcal{U}|)$: evaluating the expectation (i.e., sum over $\mathbf{x}'$) requires $|\mathcal{X}|$ operations and there are $|\mathcal{X}|$ minimizations over $|\mathcal{U}|$ possible control inputs
- **Complexity of PI per Iteration**: $O(|\mathcal{X}|^2 (|\mathcal{X}| + |\mathcal{U}|))$: the Policy Evaluation step requires solving a system of $|\mathcal{X}|$ equations in $|\mathcal{X}|$ unknowns ($O(|\mathcal{X}|^3)$), while the Policy Improvement step has the same complexity as one iteration of VI
- PI is more computationally expensive than VI
- Theoretically it takes an infinite number of iterations for VI to converge
- PI converges in $|\mathcal{U}|^{|\mathcal{X}|}$ iterations (all possible policies) in the worst case

Value Iteration

- V^* **is a fixed point of** $\mathcal{B}_*$: $V_0,\ \mathcal{B}_*[V_0],\ \mathcal{B}_*^2[V_0],\ \mathcal{B}_*^3[V_0], \ldots \quad \to V^*$

Algorithm 1 Value Iteration

Initialize V_0
for $k = 0, 1, 2, \ldots$ **do**
    $V_{k+1} = \mathcal{B}_* [V_k]$

- Q^* **is a fixed point of** $\mathcal{B}_*$: $Q_0,\ \mathcal{B}_*[Q_0],\ \mathcal{B}_*^2[Q_0],\ \mathcal{B}_*^3[Q_0], \ldots \quad \to Q^*$

Algorithm 2 Q-Value Iteration

Initialize Q_0
for $k = 0, 1, 2, \ldots$ **do**
    $Q_{k+1} = \mathcal{B}_* [Q_k]$

Policy Iteration

- Policy Evaluation: $V_0,\ \mathcal{B}_\pi[V_0],\ \mathcal{B}_\pi^2[V_0],\ \mathcal{B}_\pi^3[V_0],\ldots \quad \to V^\pi$

Algorithm 3 Policy Iteration

Initialize V_0

for $k = 0, 1, 2, \ldots$ **do**

$\quad \pi_{k+1}(\mathbf{x}) = \arg\min_{\mathbf{u}\in\mathcal{U}(\mathbf{x})} H[\mathbf{x}, \mathbf{u}, V_k(\cdot)]$ ▷ Policy Improvement

$\quad V_{k+1} = \mathcal{B}^{\infty}_{\pi_{k+1}}[V_k]$ ▷ Policy Evaluation

---

- Policy Q-Evaluation: $Q_0,\ \mathcal{B}_\pi[Q_0],\ \mathcal{B}_\pi^2[Q_0],\ \mathcal{B}_\pi^3[Q_0],\ldots \quad \to Q^\pi$

Algorithm 4 Q-Policy Iteration

Initialize Q_0

for $k = 0, 1, 2 \ldots$ **do**

$\quad \pi_{k+1}(\mathbf{x}) = \arg\min_{\mathbf{u}\in\mathcal{U}(\mathbf{x})} Q_k(\mathbf{x}, \mathbf{u})$ ▷ Policy Improvement

$\quad Q_{k+1} = \mathcal{B}^{\infty}_{\pi_{k+1}}[Q_k]$ ▷ Policy Evaluation

---

Generalized Policy Iteration

Algorithm 5 Generalized Policy Iteration

Initialize V_0
for $k = 0, 1, 2, \ldots$ **do**
    $\pi_{k+1}(\mathbf{x}) = \arg\min_{\mathbf{u} \in \mathcal{U}(\mathbf{x})} H[\mathbf{x}, \mathbf{u}, V_k(\cdot)]$ ▷ Policy Improvement
    $V_{k+1} = \mathcal{B}^n_{\pi_{k+1}}[V_k]$, for $n \geq 1$ ▷ Policy Evaluation

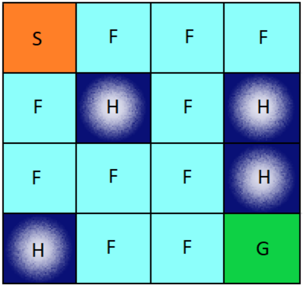
Algorithm 6 Generalized Q-Policy Iteration

Initialize Q_0
for $k = 0, 1, 2, \ldots$ **do**
    $\pi_{k+1}(\mathbf{x}) = \arg\min_{\mathbf{u} \in \mathcal{U}(\mathbf{x})} Q_k(\mathbf{x}, \mathbf{u})$ ▷ Policy Improvement
    $Q_{k+1} = \mathcal{B}^n_{\pi_{k+1}}[Q_k]$, for $n \geq 1$ ▷ Policy Evaluation

Example: Frozen Lake Problem

- Winter is here
- You and your friends were tossing around a frisbee at the park when you made a wild throw that left the frisbee out in the middle of the lake
- The water is mostly frozen but there are a few holes where the ice has melted
- If you step into one of those holes, you fall into the freezing water
- There is an international frisbee shortage so it is absolutely imperative that you navigate across the lake and retrieve the disc
- However, the ice is slippery so you cannot always move in the direction you intend

Example: Frozen Lake Problem



- S : starting point, safe
- F : frozen surface, safe
- H : hole, fall to your doom
- G : goal, where the frisbee is located
- $\mathcal{X} = \{0, 1, \ldots, 15\}$
- $\mathcal{U} = \{\text{Left}(0), \text{Down}(1), \text{Right}(2), \text{Up}(3)\}$
- You receive a reward of 1 if you reach the goal, and zero otherwise

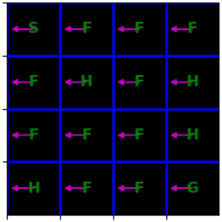
- An input $u \in \mathcal{U}$ succeeds 80% of the time. A neighboring control is executed in the other 50% of the time due to slip, e.g.,

$$x' \mid x = 9, u = 1 = \begin{cases} 13, & \text{with prob. } 0.8 \\ 8, & \text{with prob. } 0.1 \\ 10, & \text{with prob. } 0.1 \end{cases}$$

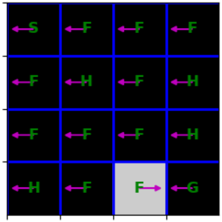
- The state remains unchanged if a control leads outside of the map

- An episode ends when you reach the goal or fall in a hole

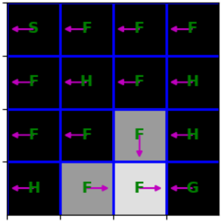
Value Iteration on Frozen Lake



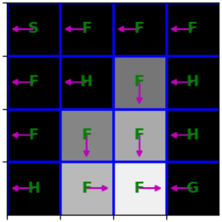
(a) $t = 0$



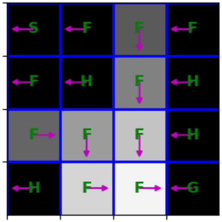
(b) $t = 1$



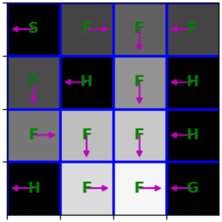
(c) $t = 2$



(d) $t = 3$



(e) $t = 4$

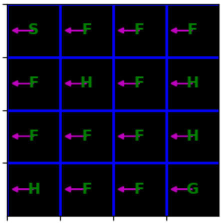


(f) $t = 5$

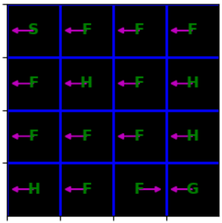
Value Iteration on Frozen Lake

Iteration	$\max_x \lvert V_{t+1}(x) - V_t(x) \rvert$	**# changed actions**	$V(0)$
0	0.80000	0	0.000
1	0.60800	1	0.000
2	0.51984	2	0.000
3	0.39508	2	0.000
4	0.30026	2	0.000
5	0.25355	2	0.254
6	0.10478	1	0.345
7	0.09657	0	0.442
8	0.03656	0	0.478
9	0.02772	0	0.506
10	0.01111	0	0.517
11	0.00735	0	0.524
12	0.00310	0	0.527
13	0.00190	0	0.529
14	0.00083	0	0.530
15	0.00049	0	0.531
16	0.00022	0	0.531
17	0.00013	0	0.531
18	0.00006	0	0.531
19	0.00003	0	0.531

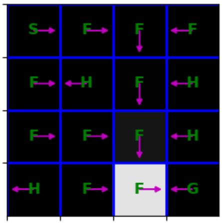
Policy Iteration on Frozen Lake



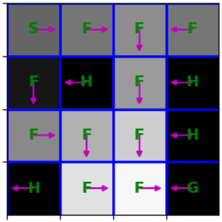
(a) $t = 0$



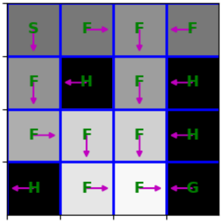
(b) $t = 1$



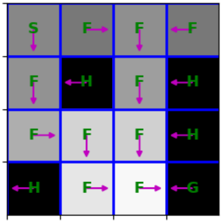
(c) $t = 2$



(d) $t = 3$



(e) $t = 4$

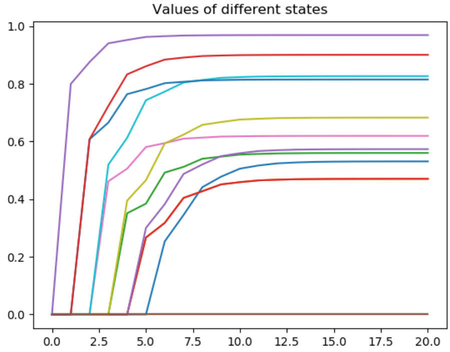


(f) $t = 5$

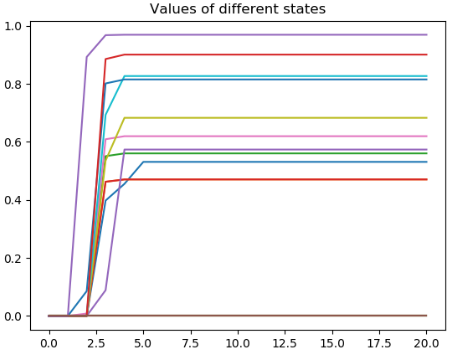
Policy Iteration on Frozen Lake

Iteration	$\max_x \lvert V_{t+1}(x) - V_t(x) \rvert$	**# changed actions**	$V(0)$
0	0.00000	0	0.000
1	0.89296	1	0.000
2	0.88580	9	0.398
3	0.48504	2	0.455
4	0.07573	1	0.531
5	0.00000	0	0.531
6	0.00000	0	0.531
7	0.00000	0	0.531
8	0.00000	0	0.531
9	0.00000	0	0.531
10	0.00000	0	0.531
11	0.00000	0	0.531
12	0.00000	0	0.531
13	0.00000	0	0.531
14	0.00000	0	0.531
15	0.00000	0	0.531
16	0.00000	0	0.531
17	0.00000	0	0.531
18	0.00000	0	0.531
19	0.00000	0	0.531

Value Iteration vs Policy Iteration

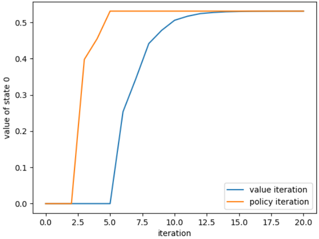


(a) VI

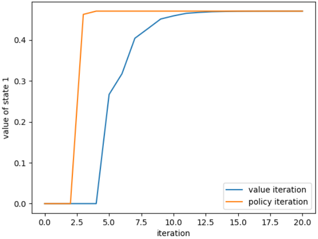


(b) PI

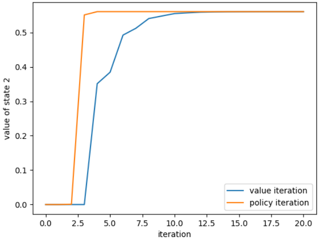
Value Iteration vs Policy Iteration



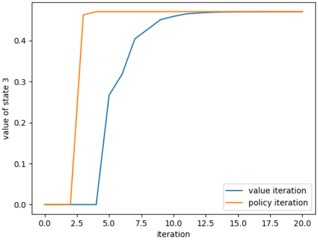
(a) State 0



(b) State 1



(c) State 2



(d) State 3

Outline

Linear Programming Solution to the Bellman Equation

- Consider a Discounted Problem with finite state space $\mathcal{X}$
- Suppose we initialize VI with V_0 that satisfies a relaxed Bellman equation condition:

$$V_0(\mathbf{x}) \leq \min_{\mathbf{u} \in \mathcal{U}(\mathbf{x})} \left(\ell(\mathbf{x}, \mathbf{u}) + \gamma \sum_{\mathbf{x}' \in \mathcal{X}} p_f(\mathbf{x}' \mid \mathbf{x}, \mathbf{u}) V_0(\mathbf{x}') \right), \qquad \forall \mathbf{x} \in \mathcal{X}$$

- Since $\mathcal{B}_*$ is monotone, applying VI to V_0 leads to:

$$\begin{aligned}
V_1(\mathbf{x}) &= \min_{\mathbf{u} \in \mathcal{U}(\mathbf{x})} \left(\ell(\mathbf{x}, \mathbf{u}) + \gamma \sum_{\mathbf{x}' \in \mathcal{X}} p_f(\mathbf{x}' \mid \mathbf{x}, \mathbf{u}) V_0(\mathbf{x}') \right) \geq V_0(\mathbf{x}), \quad \forall \mathbf{x} \in \mathcal{X} \\
V_2(\mathbf{x}) &= \min_{\mathbf{u} \in \mathcal{U}(\mathbf{x})} \left(\ell(\mathbf{x}, \mathbf{u}) + \gamma \sum_{\mathbf{x}' \in \mathcal{X}} p_f(\mathbf{x}' \mid \mathbf{x}, \mathbf{u}) V_1(\mathbf{x}') \right) \\
&\geq \min_{\mathbf{u} \in \mathcal{U}(\mathbf{x})} \left(\ell(\mathbf{x}, \mathbf{u}) + \gamma \sum_{\mathbf{x}' \in \mathcal{X}} p_f(\mathbf{x}' \mid \mathbf{x}, \mathbf{u}) V_0(\mathbf{x}') \right) = V_1(\mathbf{x}), \quad \forall \mathbf{x} \in \mathcal{X}
\end{aligned}$$

Linear Programming Solution to the Bellman Equation

- The above shows that $V_{k+1}(\mathbf{x}) \geq V_k(\mathbf{x})$ for all k and $\mathbf{x} \in \mathcal{X}$
- Since VI guarantees that $V_k(\mathbf{x}) \to V^*(\mathbf{x})$ as $k \to \infty$, we also have:

$$V^*(\mathbf{x}) \geq V_0(\mathbf{x}), \quad \forall \mathbf{x} \in \mathcal{X} \quad \Rightarrow \quad \sum_{\mathbf{x} \in \mathcal{X}} w(\mathbf{x}) V^*(\mathbf{x}) \geq \sum_{\mathbf{x} \in \mathcal{X}} w(\mathbf{x}) V_0(\mathbf{x})$$

for any $w(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{X}$.

- The above holds for **any** V_0 that satisfies:

$$V_0(\mathbf{x}) \leq \min_{\mathbf{u} \in \mathcal{U}(\mathbf{x})} \left(\ell(\mathbf{x}, \mathbf{u}) + \gamma \sum_{\mathbf{x}' \in \mathcal{X}} p_f(\mathbf{x}' \mid \mathbf{x}, \mathbf{u}) V_0(\mathbf{x}') \right), \qquad \forall \mathbf{x} \in \mathcal{X}$$

- Since V^* satisfies this condition with equality (Bellman Equation), it is the maximal V_0 that satisfies the condition

Linear Programming Solution to the Bellman Equation

LP Solution to Bellman Equation (Discounted Problem)

For finite $\mathcal{X}$, the solution $V^*(\mathbf{x})$ to the linear program with $w(\mathbf{x}) > 0$:

$$\max_{V} \sum_{\mathbf{x}\in\mathcal{X}} w(\mathbf{x}) V(\mathbf{x})$$

$$\text{s.t.} \quad V(\mathbf{x}) \leq \left(\ell(\mathbf{x}, \mathbf{u}) + \gamma \sum_{\mathbf{x}'\in\mathcal{X}} p_f(\mathbf{x}' \mid \mathbf{x}, \mathbf{u}) V(\mathbf{x}') \right), \qquad \forall \mathbf{u} \in \mathcal{U}, \forall \mathbf{x} \in \mathcal{X}$$

also solves the Bellman Equation to yield the optimal value function of an infinite-horizon finite-state discounted stochastic optimal control problem.

- An equivalent result holds for the First-Exit Problem

LP Solution to Bellman Equation (Proof)

- Let J^* be the solution to the linear program so that:

$$J^*(\mathbf{x}) \leq \left(\ell(\mathbf{x},\mathbf{u}) + \gamma \sum_{\mathbf{x}' \in \mathcal{X}} p_f(\mathbf{x}' \mid \mathbf{x},\mathbf{u}) J^*(\mathbf{x}')\right), \qquad \forall \mathbf{u} \in \mathcal{U}, \forall \mathbf{x} \in \mathcal{X}$$

- Since J^* is feasible, it satisfies $J^*(\mathbf{x}) \leq V^*(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}$
- By contradiction, suppose that $J^* \neq V^*$
- Then, there exists a state $\mathbf{y} \in \mathcal{X}$ such that:

$$J^*(\mathbf{y}) < V^*(\mathbf{y}) \quad \Rightarrow \quad \sum_{\mathbf{x} \in \mathcal{X}} w(\mathbf{x}) J^*(\mathbf{x}) < \sum_{\mathbf{x} \in \mathcal{X}} w(\mathbf{x}) V^*(\mathbf{x})$$

  for any positive $w(\mathbf{x})$ but since V^* solves the Bellman Equation:

$$V^*(\mathbf{x}) \leq \left(\ell(\mathbf{x},\mathbf{u}) + \gamma \sum_{\mathbf{x}' \in \mathcal{X}} p_f(\mathbf{x}' \mid \mathbf{x},\mathbf{u}) V^*(\mathbf{x}')\right), \qquad \forall \mathbf{u} \in \mathcal{U}, \forall \mathbf{x} \in \mathcal{X},$$

  V^* is feasible and has higher value than J^*, which is a contradiction.

Dual Linear Program

- Dual linear program:

$$\begin{aligned} \min_{\lambda \geq 0} \quad & \sum_{\mathbf{x}\in\mathcal{X}} \sum_{\mathbf{u}\in\mathcal{U}} \ell(\mathbf{x},\mathbf{u})\lambda(\mathbf{x},\mathbf{u}) \\ \text{s.t.} \quad & \sum_{\mathbf{u}'\in\mathcal{U}} \lambda(\mathbf{x}',\mathbf{u}') = w(\mathbf{x}) + \gamma \sum_{\mathbf{x}\in\mathcal{X}} \sum_{\mathbf{u}\in\mathcal{U}} \lambda(\mathbf{x},\mathbf{u}) p_f(\mathbf{x}' \mid \mathbf{x},\mathbf{u}), \qquad \forall \mathbf{x}' \in \mathcal{X} \end{aligned}$$

- If $\sum_{\mathbf{x}\in\mathcal{X}} w(\mathbf{x}) = 1$, the constraint ensures that $\lambda(\mathbf{x},\mathbf{u})$ is a probability measure on $\mathcal{X} \times \mathcal{U}$ induced by an optimal policy π:

$$\lambda(\mathbf{x},\mathbf{u}) = \sum_{\mathbf{x}_0\in\mathcal{X}} w(\mathbf{x}_0) \sum_{t=0}^{\infty} \gamma^t \mathbb{P}^{\pi}(\mathbf{x}_t = \mathbf{x}, \mathbf{u}_t = \mathbf{u} | \mathbf{x}_0)$$

- Optimal policy:

$$\pi^*(\mathbf{x}) \in \arg\min_{\mathbf{u}\in\mathcal{U}} \lambda(\mathbf{x},\mathbf{u})$$